PRETZEL KNOTS WITH UNKNOTTING NUMBER ONE

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Abstract. We provide a partial classification of all 3-strand pretzel knots $K = P(p, q, r)$ with unknotting number one. Following the classification by both Kobayashi and Scharlemann-Thompson for all parameters odd, we treat the remaining families with $r = 2m$. We first determine that there are only four possible such families which may satisfy $u(K) = 1$. These families are determined by the sum $p + q$ as well as the sign of a certain Goeritz matrix $G(K)$, and we resolve the problem in two of these cases. Ingredients in our proofs include Donaldon's diagonalisation theorem (and Greene's strengthening thereof), Nakanishi's unknotting bounds from the Alexander module, and the correction terms introduced by Ozsváth and Szabó. Based on our results, we conjecture that the only 3-stranded pretzel knots $P(p, q, r)$ with unknotting number one that are not 2-bridge knots are $P(3, -3, 2)$ and its reflection.

1. Introduction

The unknotting number is the minimal number of times a knot must be passed through itself in order to unknot it. Although it is simple to understand, in most cases it is extremely difficult to compute. Indeed, the majority of knots with as few as 12 crossings still have unknown unknotting numbers. While exhibiting an upper bound is straightforward (by actually performing an unknotting), lower bounds have been difficult to achieve: it is generally not known which knot diagrams will realize the actual unknotting number (see [1], [17], and [26]).

Our main result here is a partial classification of 3-strand pretzels $K = P(p, q, r)$ with unknotting number one. Such knots are unchanged by any permutation of the parameters, and the reflection is given by

$$\overline{P(p, q, r)} = P(-p, -q, -r)$$

Observe that as $u(\overline{K}) = u(K)$, our classification need only be up to reflection. For K to be a *bona fide* knot, we require either all three parameters odd, or exactly one even (say $r = 2m$). The first case has been studied by Kobayashi [9], and independently by Scharlemann and Thompson [24], who give the criterion that

$$u(K) = 1 \iff \pm\{1, 1\} \text{ or } \pm\{3, -1\} \subset \{p, q, r\}$$

and thus our work focuses on the case $P(p, q, 2m)$. As a consequence of the reflection invariance in unknotting number, we assume that $2m$ is positive.

Kanenobu and Murakami, [8], and Torisu, [28], have given a complete description of the 2-bridge knots with unknotting number one. After observing that the double branched cover of a pretzel is Seifert fibred over S^2, we notice that a pretzel knot $P(p, q, r)$ is not a 2-bridge knot if and only if all three of $p, q, r \neq \pm 1$ (else we have fewer than three exceptional fibres, and hence a lens space). When $r = 2m$ for $m \in \mathbb{Z}$, then $r \neq \pm 1$, and therefore we rule out the cases $p, q \neq \pm 1$.

1.1. Main Results.

We first determine that there are only four possible families of 3-stranded pretzel knots (excluding 2-bridge knots) with r even, which may have unknotting number one. These families are determined by the sum $p+q$ as well as the sign of a certain Goeritz matrix $G(K)$. Our main theorem is then the following.

Main Theorem. *Suppose that $K = P(p, q, r)$, $r = 2m$ even, is a pretzel with unknotting number one. Then, up to reflection, $p + q = 0, \pm 2, 4$ and $m > 0$. Moreover:*

1. *If $p + q = -2$, then $K = P(1, -3, 2m)$ (all 2-bridge);*
2. *If $p + q = 0$, then $K = P(3, -3, 2)$ (which is not 2-bridge).*

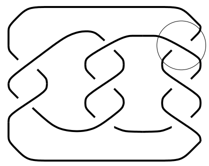
The table below indicates which pretzels in each family have unknotting number one, together with our conjectures. We present it as a more digestible version of the theorem's conclusions, but ask that the reader remember that the families were isolated by considering the pretzel up to reflection.

Family	Knots must be...	Conjecture
$p+q=-2$	$P(1,-3,2m)$	–
$p+q=2$	?	$P(3,-1,2m), P(1,1,2m)$
$p+q=4$	?	$P(3,1,2), P(5,-1,4), P(5,-1,2)$
$p+q=0$	$P(3,-3,2)$	

Most of these are in fact 2-bridge, as at least one parameter is ± 1. Hence, we have the following conjecture:

Conjecture 1.1. *The only 3-stranded pretzel knots $P(p,q,r)$ with unknotting number one that are not 2-bridge knots are $P(3,-3,2)$ and its reflection.*

The pretzel referred to in this conjecture is the following:



and the circle indicates the unknotting crossing.

2. Motivation

A classical lower bound for the unknotting number is given by the knot signature, $\sigma(K)$, which obeys the inequality $|\sigma(K)| \leq 2u(K)$ (see [15]). Thus if $u(K)=1$, then $|\sigma(K)|=0,2$. More recently Rasmussen's s-invariant, $s(K)$, has been shown to give a lower bound on the unknotting number of a knot (see [23]): $|s(K)| \leq 2u(K)$.

Specific to the case of unknotting number one, there are a number of topological obstructions, many concerning the double branched cover $\Sigma(K)$. The most important for this paper is Montesinos' theorem: if $u(K)=1$, then $\Sigma(K)$ arises as half-integral surgery on some knot $\kappa \subset S^3$. That is, $\Sigma(K) = S^3_{\pm D/2}(\kappa)$ (see [13]). This has various implications: cyclic $H_1(\Sigma(K))$, restrictions on the 4-manifolds with $\Sigma(K)$ as boundary, and symmetries in the correction terms (see [19]) of $\Sigma(K)$.

This work was motivated by the following question: Which algebraic knots, in the sense of Conway, have unknotting number equal to one? A complete treatment of algebraic knots can be found in [5] and [27], but in brief, the distinct types are 2-bridge, large algebraic, and Montesinos length 3, with the characterisation being split according to the topology of their double covers. To wit, we have the following division.

K	2-bridge	large algebraic	Montesinos length 3
$\Sigma(K)$	lens space	graph manifold (toroidal)	atoroidal Seifert fibred (S^2 with 3 exceptional fibres)

As stated above, Kanenobu and Murakami have solved the problem for 2-bridge knots in [8], and this solution was later generalised using Gordian distance by Torisu, [28]. The large algebraic case is dealt with by Gordon and Luecke in [5], in terms of the constituent algebraic tangles of K. However, the double branched cover of a Montesinos knot (length 3) is neither a lens space nor toroidal, so neither of these results apply. It is then natural to ask the following question.

Question 2.1. *Which Montesinos knots of length three have unknotting number one?*

In [28] Torisu makes the following conjecture.

Conjecture 2.2 (Torisu)**.** *Let K be a Montesinos knot of length three. Then $u(K)=1$ if and only if $K=\mathcal{M}(0;(p,r),(q,s),(2mn\pm 1, 2n^2))$, where p, q, r, s, m, and n are non-zero integers, m and n are coprime, and $ps+rq=1$.*

Torisu also proves the following theorem about the unknotting number of pretzel knots of length three.

Theorem 2.3 (Torisu, [28])**.** *Let K be a Montesinos knot of length three and suppose the unknotting operation is realized in a standard diagram. Then $u(K)=1$ if and only if*

$$K=\mathcal{M}(0;(p,r),(q,s),(2mn\pm 1,2n^2))$$

where p, q, r, s, m, and n are non-zero integers, m and n are coprime, and $ps+rq=1$.

It is easy to show that the second condition implies the first, however it is not known if every Montesinos knot of length three with unknotting number one can be written in the described form. Nevertheless a proof of the following conjecture would also prove Conjecture 2.2.

Conjecture 2.4 (Seifert fibring conjecture)**.** *For a knot in S^3 which is neither a torus knot nor a cable of a torus knot, only integral surgery slopes can yield a Seifert fibred space.*

A complete explanation of why Conjecture 2.4 implies Conjecture 2.2 can be found in [28]. In short, if Montesinos knot $K \subset S^3$ has unknotting number one, then $\Sigma(K)$, a Seifert fibred space, equals $S^3_{\pm D/2}(\kappa)$, where D is odd and $\kappa \subset S^3$ is a knot. If the Seifert fibering conjecture is true, then κ is either a torus knot or a cable of a torus knot. In either case, Dehn surgery on these knots is well understood (see Moser, [14]), and after some calculations the desired result is achieved.

Of interest to us is what Torisu's conjecture predicts about 3-stranded pretzel knots with unknotting number equal to one. Indeed, it gives all the results we have proved, and supports our conjectured result in the $p + q = 2$ case. Thus, our work can be seen as a partial proof of Torisu's conjecture.

2.1. **Organisation.** We first use the signature and Rasmussen's invariant to separate our knots into two families of candidates, one being slice, the other not. Both require different approaches.

For the non-slice family, we use the Montesinos theorem coupled with certain plumbings for $\Sigma(K)$ to glue together a closed, oriented, simply connected, smooth, negative-definite 4-manifold, and thence apply Donaldson's diagaonalisation theorem. This turns out to be insufficient as an obstruction to unknotting number one, so to make more progress we use a strengthened version of this approach due to Greene. The result is, in the case $p + q = -2$, that K must be 2-bridge to satisfy $u(K) = 1$. We conjecture that this is in general true (i.e. for the remaining $p + q = 2, 4$ case).

In the slice case, we do two things. First, we use a theorem of Nakanishi on the Alexander module of the knot to narrow down the value of m to 1 (up to sign). Then, the correction terms of $\Sigma(K)$, as defined by Ozsváth and Szabó, can be used to prove that $p = 3$. Along the way, we weaken the Ozsváth-Szabó obstruction to $u(K) = 1$, so that we might narrow down possible Spin^c-structure labellings compatible with the required symmetries, and then use the full obstruction to complete the proof in these restricted cases.

Our results give us evidence for the truth of our conjecture, which would leave only the chiral knot $P(3, -3, 2)$ and its reflection as the non-2-bridge knots with having unknotting number one.

2.2. **Acknowledgements.** The authors would like to extend their thanks to Cameron Gordon, Josh Greene, Raymond Lickorish, and Andrew Lobb for helpful discussions, and to Ana Lecuona and Brendan Owens for their careful reading of preliminary versions. DB is supported in part by EPSRC Grants EP/H0313671, EP/G0395851 and EP/J1075308, and thanks the LMS for their Scheme 2 Grant. JG is supported by the Rector's Award and SOF at Imperial College London. ES is partially supported by NSF RTG Grant DMS-0636643.

3. Signature Requirements

We use the following theorem to determine the signature of a knot, which is Theorem 6 in [4]:

Theorem 3.1 (Gordon-Litherland)**.** *For any regular projection K of the knot k, with associated Goeritz matrix $G(K)$,*

$$\sigma(k) = \operatorname{sgn}(G(K)) - \mu(K),$$

where μ is the correction term of K which depends on the crossings of K and a spanning surface of K.

As a brief note before continuing, when we speak of the determinant of K, this will always be positive. The determinant of $G(K)$, however, can be signed, and this is important for our later classification. Thus, in general, $\det K = |\det G(K)|$.

That said, we apply the theorem above to a standard diagram of the knot $P(p, q, r)$, Figure 1(a), where p and q are odd, and r is even. By shading and labeling the three regions of Figure 1(a) as marked by the X_i in 1(b), we obtain the Goeritz matrix of K:

$$G(K) = \begin{pmatrix} p+r & -p \\ -p & p+q \end{pmatrix}.$$

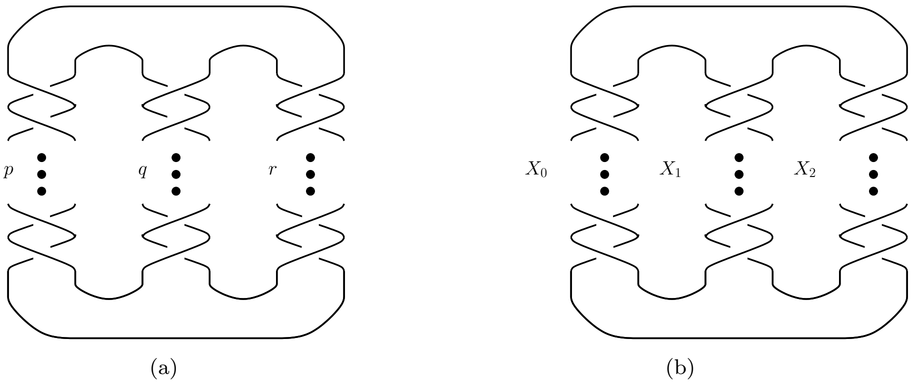


Figure 1. (a) Pretzel knot $P(p,q,r)$, where $p<0$, and $q,r>0$; and (b) the knot $P(p,q,r)$ with a checkerboard coloring

Note that the matrix $G(K)$ is 2×2, and therefore $\mathrm{sgn}(G(K)) \in \{-2,0,2\}$. In particular if $u(K)=1$, then μ is restricted to $\{-4,-2,0,2,4\}$. According to [4], the correction term μ is the sum of the crossing numbers of the strands along the p and q strands. Since $|p|,|q|,|r|>1$, if p and q are both positive then $\mu = p+q \geq 6$, a contradiction. So without loss of generality, take $p>0$ and $q<0$. Furthermore the reflection invariance of unknotting number allows us to assume $r=2m>0$ (we ignore $m=0$ for reasons below). Relabel the knot $K=P(p,q,r)$ as $K=P(k,-k+n,2m)$, where $m>0$, $k>0$ odd, and $n\in\{-4,-2,0,2,4\}$. The Goeritz matrix thus becomes:

$$G(K) = \begin{pmatrix} k+2m & -k \\ -k & n \end{pmatrix},$$

which implies

$$\det(G(K)) = -k^2+kn+2mn$$

If we consider the case when $n=-4$, then we compute easily that $\sigma(K)=4$, which is not within the range for unknotting number one. Also, if $n=4$ and $\det G(K)<0$, then $\sigma(K)=-4$, and we can rule this possibility out for the same reason. Hence, we have five remaining cases:

Case	n	$\det G(K)$	$\sigma(K)$
1	-2		2
2a	2	<0	-2
2b	2	>0	0
3	4	>0	-2
4	0		0

All except the last are not slice knots (Jabuka and Lecuona [11], c.f. Greene [7]), whereas all the pretzels in the last are slice (ibid.). Since we have different methods for the slice and non-slice families, Section 4 first considers Cases 1-3, when the knots are not slice, and Sections 5 onwards consider those that are slice.

For completeness, we can treat the only ignored case, $m=0$, immediately. In this instance, it is clear that $P(p,q,0)=T(p,2)\#T(q,2)$, and then since unknotting number one knots are prime (see Scharlemann, [25], and Zhang, [29]), it follows that one of $p,q=\pm 1$. Then the Milnor Conjecture, in Kronheimer and Mrowka, [10], and Rasmussen, [23], gives the following result, which justifies our ignoring $m=0$ in the above theorem.

Lemma 3.2. *If $K=P(p,q,0)$ and $u(K)=1$, for p,q odd, then $pq=\pm 3$.*

4. Non-Slice Case

In this section we consider the case of non-slice 3-stranded pretzel knots of the form $P(k,-k+n,2m)$ for $n=\pm 2,4$, where k odd, $k\geq 1$, and $m>0$. Our method has three main ingredients: the signed Montesinos theorem, Donaldson's diagonalisation theorem, and Greene's strengthening of Donaldson's theorem for L-spaces. Our ultimate goal would be to prove the following conjecture.

Conjecture 4.1. *Suppose $K=P(p,q,r)$ is a non-slice 3-stranded pretzel knot with $u(K)=1$. Then K is 2-bridge.*

4.1. **The case** $n = -2$**.** Recall that knots of the form $K = P(k, -k-2, 2m)$ have signature 2. To avoid knots which are 2-bridge we assume $k \geq 3$. First consider the "signed" version of the Montesinos theorem (see Proposition 4.1, [6]).

Theorem 4.2 (Signed Montesinos)**.** *Suppose that K is a knot that is undone by changing a negative crossing (so $\sigma(K) = 0, 2$). Then $\Sigma(K) = S^3_{-\epsilon D/2}(\kappa)$ for some knot $\kappa \subset S^3$, where $D = \det(K)$, and $\epsilon = (-1)^{\frac{1}{2}\sigma(K)}$. In particular, $-\Sigma(K) = S^3_{\epsilon D/2}(\overline{\kappa})$ bounds a smooth, simply connected, 4-manifold W_K with ϵ-definite intersection form $-\epsilon R_n$, where*

$$R_n = \begin{pmatrix} -n & 1 \\ 1 & -2 \end{pmatrix}$$

and $D = 2n - 1$.

If $u(K) = 1$, in our case we have $\sigma(K) = 2$, and so $-\Sigma(K)$ bounds a negative-definite 4-manifold W_K from Theorem 4.2. In order to use Donaldson's Theorem A we need another 4-manifold which is bounded by $\Sigma(K)$, call this X_K, with intersection form Q_K, so that we can glue them together to obtain a closed manifold $X = X_K \cup_{\Sigma(K)} W_K$. Since the boundary is a rational homology 3-sphere, the intersection form is given by $Q_X = Q_K \oplus R_n$. As W_K is simply connected (obtained from a surgery diagram), if X_K is simply connected then so too is X. We are now ready to use Donaldson's Theorem A.

Theorem 4.3 (Donaldson, [3])**.** *Let X be a closed, oriented, simply connected, smooth 4-manifold. If the intersection form Q_X is negative definite, then there exists an integral matrix A such that $-AA^T = Q_X$.*

Thus, if we can show that there does not exist an integral matrix A such that $-AA^T = Q_X$, the knot $K = P(k, -k-2, 2m)$ must have unknotting number greater than one. The first question, then, is how to find X, and for this we use plumbing.

Plumbings: Let G be a vertex-weighted simple graph, with vertices v labelled $w(v)$. In general, we take $w(v) < 0$ since we are mainly concerned with negative definite manifolds. To construct a 4-manifold $X = X(G)$ from G, take, for every vertex v, the 2-disc bundle $B(v)$ over S^2 of Euler number $w(v)$, and plumb $B(v)$ and $B(v')$ if and only if v and v' are adjacent in G. This manifold X has free $H_2(X)$, generated by the homology classes $[v]$ corresponding to the vertices.

Supposing that G is a tree, then $X(G)$ is simply connected. The manifold $Y = Y(G) = \partial X$ is given by a Kirby diagram of unknots, with linking matrix given by the adjacency matrix for G. The intersection form Q for $X(G)$ is then also the matrix for G. Explicitly, we have $[v] \cdot [v] = w(v)$ for each vertex, and $[v] \cdot [v'] = 1$ if the two distinct vertices are connected by an edge, zero otherwise.

Since $\Sigma(K)$ is a Seifert fibred space, it has a surgery presentation given by unknots with linking matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & p & 0 & 0 \\ 1 & 0 & q & 0 \\ 1 & 0 & 0 & r \end{pmatrix}$$

Hence, we can obtain a plumbing with boundary $\Sigma(K)$ using the graph with this matrix. However, for what will follow, this 4-manifold is insufficient, so we use the work of Neumann and Raymond in [18] to construct a related manifold. First, we find a continued fraction expansion for $p/(p-1)$, $q/(q-1)$, $r/(r-1)$:

$$\frac{p}{p-1} = [p_1, p_2, \ldots, p_i] \qquad \frac{q}{q-1} = [q_1, q_2, \ldots, q_j] \qquad \frac{r}{r-1} = [r_1, r_2, \ldots, r_k],$$

where

$$[x_1, x_2, \ldots, x_n] = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\ddots - \frac{1}{x_n}}}$$

We then let $\widetilde{G}$ be the weighted graph as in Figure 2, and let X'_K be the 4-manifold obtained by plumbing $\widetilde{G}$.

This is significant for the following reason (c.f. Greene-Jabuka, [7], Lemma 3.2).

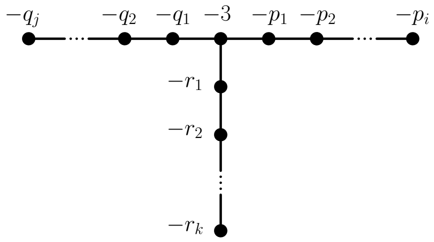


Figure 2. A weighted graph $\widetilde{G}$.

Lemma 4.4 (Neumann and Raymond)**.** *The incident matrix of the weighted graph $\widetilde{G}$ from figure 2 is negative definite if and only if p, q, and r satisfy*

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>0$$

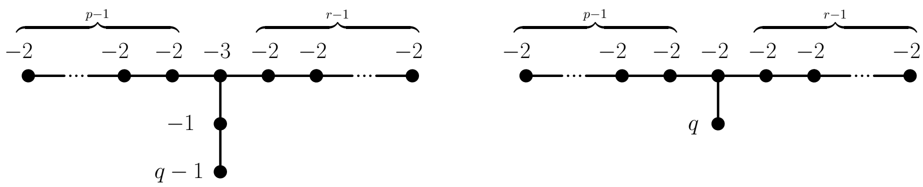
Corollary 4.5. *Let $K = P(k, -k-2, 2m)$. The 4-manifold determined by the graph $\widetilde{G}$ is negative definite.*

Proof. The condition rearranges to $\frac{\det G(K)}{pqr} > 0$, and as only $q < 0$, we see that $\det G(K) < 0$. But $\det(G(K)) = -k^2 + kn + 2mn = -k^2 - 2k - 4m$, which is clearly negative. $\square$

Applying this method to our particular pretzel, we get continued fraction expansions of

$$\frac{p}{p-1} = \overbrace{[2,2,\ldots,2]}^{p-1} \qquad \frac{q}{q-1} = \overbrace{[2,2,\ldots,2]}^{q-1} \qquad \frac{r}{r-1} = [1,-r+1].$$

We then obtain $\widetilde{G} = \widetilde{G}(p,q,r)$, as in Figure 3(a). By blowing down the -1 framed vertex of $\widetilde{G}$ we obtain the graph $G = G(p,q,r)$ as in Figure 3(b).



(a) (b)

Figure 3. (a) The weighted graph $\widetilde{G}(p,q,r)$; and (b) a weighted graph $G(p,q,r)$

Let X_K be the manifold described by Figure 3(b). The manifolds represented in these two figures have identical boundary as the second is obtained from the first by a blow-up on the -1 weighted vertices, which corresponds to $+1$ Dehn twists around the corresponding unknots in the surgery diagram for the boundary. Moreover, since we can handle-slide the -1 vertex away from the rest of the graph, by blowing it up we are removing a negative eigenvalue, which in turn means that both the 4-manifold and its blow-up with be negative definite together.

Form the closed, smooth, oriented manifold $X = X_K \cup_{\Sigma(K)} W_K$. Unfortunately, for this X, there always exists an A such that $-AA^T = Q_X$ for any odd k and positive m. However, in [6], Greene uses Heegaard Floer homology to strengthen this approach, and we too will use this stronger version of Donaldson's theorem to achieve our result. In order to explain this strengthening, we review some of the theory of Heegaard Floer homology.

Correction Terms and Sharpness: Ozsváth and Szabó have also shown, in [19], that the Heegaard Floer homology of a rational homology sphere Y is absolutely graded over $\mathbb{Q}$. They also give a definition of correction terms, $d(Y, \mathfrak{t})$ which are the minimally graded non-zero part in the image of $HF^\infty(Y, \mathfrak{t})$ inside $HF^+(Y, \mathfrak{t})$. These are strongly connected to the topology of 4-manifolds with Y as boundary, for any such negative definite smooth oriented X which has an $\mathfrak{s} \in \mathrm{Spin}^c(X)$ such that $\mathfrak{s}|_Y = \mathfrak{t}$ must satisfy

$$\mathfrak{c}_1(\mathfrak{s})^2 + b_2(X) \leq 4d(Y, \mathfrak{t}) \tag{1}$$

A rational homology 3-sphere Y is an *L-space* if rank $\widehat{HF}(Y) = |H_1(Y)|$. Furthermore a *sharp* manifold is defined by Ozsváth and Szabó as follows.

Definition 4.6. *A negative-definite smooth 4-manifold X with L-space boundary Y is* sharp *if, for every $\mathfrak{t} \in Spin^c(Y)$, there is some $\mathfrak{s} \in Spin^c(X)$ with $\mathfrak{s}|_Y = \mathfrak{t}$ that attains equality in the bound (1).*

We are now able to present Greene's theorem.

Theorem 4.7 (Greene)**.** *Suppose K is a knot in S^3, $\Sigma(K)$ is an L-space, and $u(K) = 1$. Suppose also that either $\sigma(K) = 0$ and K is undone by changing a positive crossing, or that $\sigma(K) = 2$. If X_K is a smooth, sharp, simply connected 4-manifold with rank r negative-definite intersection form Q_K, and X_K is bounded by $\Sigma(K)$, then there exists an integral matrix A such that $-AA^T = Q_K \oplus R_n$, and A can be chosen such that the last two rows are $(0, 1, x_3, \ldots, x_{r+2})$ and $(1, -1, 0, \ldots, 0)$. Furthermore the values $x_3, \ldots, x_{r+2}$ are non-negative integers and obey the condition*

$$x_3 \leq 1,\ x_i \leq x_3 + \cdots + x_{i-1} + 1 \ \textit{for } 3 < i < r + 2 \tag{2}$$

and the upper right $r \times r$ matrix of A has determinant ± 1.

We have already shown tha X_K is simply connected and negative-definite, so what remains is to check that $\Sigma(K)$ is an L-space, and X_K is sharp. For the L-space condition, we quote the following theorem.

Theorem 4.8 (Champanerkar-Kofman [2])**.** *Let $L = P(p_1, -q, p_2)$ with $p_1, p_2, q \geq 2$. The space $\Sigma(L)$ is and L-space if and only if at least one of the following hold:*

(1) $q \geq min\{p_1, p_2\}$,
(2) $q = min\{p_1, p_2\} - 1$ *and* $max\{p_1, p_2\} \leq 2p + 1$.

Furthermore, the theorem is unchanged for all permutations of p_1, p_2, q, and can be altered in the obvious way for links $P(-p_1, -p_2, q)$.

Corollary 4.9. *Let $K = P(k, -k-2, 2m)$ as above. Then $\Sigma(K)$ is an L-space.*

Proof. Direct application of the above theorem. $\square$

Finally to show the manifold X_K is sharp we use Theorem 1.5 in [20]. Since the negative definite plumbing diagram has one "bad" vertex, in the sense of [20], it follows that X_K is sharp.

Now we are (finally) ready to show that the A described in Theorem 4.7 does not exist. We begin by writing down the intersection form of the manifold X:

$$Q_X = \left(\begin{array}{cccccccc|cc}
-2 & 1 & 0 & & \cdots & & & 0 & & \\
1 & -2 & 1 & & & & & & & \\
0 & 1 & -2 & & & & & & & \\
 & & & & \ddots & & & 1 & & \\
\vdots & & & \ddots & \ddots & & & & & \\
 & & & & & & 1 & & & \\
 & & & & & 1 & -2 & 0 & & \\
0 & & & 1 & & & 0 & -k-2 & & \\
\hline
 & & & & & & & & -n & 1 \\
 & & & & & & & & 1 & -2
\end{array}\right)$$

where $M_{k+2m,k} = M_{k,k+2m} = 1$. It will be helpful to label the rows of A as $v_1, \ldots, v_{k+2m}$. Observe that $M_{i,j} = -(AA^T)_{i,j} = -v_i \cdot v_j$. Since $|v_i \cdot v_i| = 2$ for $i \neq k+2m$ and $k+2m+1$, each row of A except row $k + 2m$ and row $k + 2m + 1$ has two nonzero entries, each of magnitude 1. Without loss of generality set $v_1 = (1, -1, 0, \ldots, 0)$. Making this choice, along with the two row conditions from Theorem 4.7, the remaining rows must take the following form.

$$
A = \left(\begin{array}{cccccc|cc}
1 & -1 & & & & & & \\
 & 1 & -1 & & & & & \\
 & & \ddots & \ddots & & & & \\
 & & & & & & & \\
 & & & & 1 & -1 & & \\
* & * & \dots & \dots & * & * & a & b \\
\hline
* & * & \dots & \dots & * & * & & 1 \\
 & & & & & & 1 & -1
\end{array}\right)
$$

Next let $A_{k+2m+1,1} = \alpha$. Since $v_i \cdot v_{i+1} = 1$ for $i = 1, 2, \dots, k+2m-2$, each of the first $k+2m$ entries along the $k+2m$ row all equal α.

$$
A = \left(\begin{array}{cccccc|cc}
1 & -1 & & & & & & \\
 & 1 & -1 & & & & & \\
 & & \ddots & \ddots & & & & \\
 & & & & & & & \\
 & & & & 1 & -1 & & \\
* & * & \dots & \dots & * & * & a & b \\
\hline
\alpha & \alpha & \dots & \dots & \alpha & \alpha & & 1 \\
 & & & & & & 1 & -1
\end{array}\right)
$$

According to Theorem 4.7, $\alpha = 0$ or 1.

(1) If $\alpha = 0$, $v_{k+2m+1}^2 = n = \frac{1}{2}(\det(K) + 1) = 1$, and therefore $\det(K) = 1$. One can use the Goeritz matrix to show that $\det P(k, -k-2, 2m) = k^2 + 2k + 4m$, so clearly $\alpha \neq 0$.

(2) If $\alpha = 1$, then $v_{k+2m+1}^2 = n = \frac{1}{2}(\det K + 1) = k + 2m + 1$. This only happens if $k^2 = 1$, which contradicts our assumption that $k \geq 3$.

The reader will note that knots of the form $P(1, -3, 2m)$ have unknotting number one for all integral m. We have thus proved the following, the first piece of our classification.

Theorem 4.10. *Suppose that $P(k, -k-2, 2m)$ has unknotting number one, $k, m > 0$, k odd. Then $k = 1$.*

4.2. **The cases $n = 2, 4$.** In these cases, we run into some difficulties applying the above method. First of all, we cannot employ Greene's theorem, since $\Sigma(K)$ is not an L-space in this instance. Second, it is conceivable that Donaldson's theorem is sufficient by itself, but the manifolds obtained by plumbing diagonalise, so we do not get an obstruction that way. And finally, the two techniques that follow below fail as well: Nakanishi's method yields $u(K) \geq 1$, and the Ozsváth-Szabó correction term symmetry is not applicable (again because of the L-space condition).

What we can do, though, with support from Torisu's conjecture, is hypothesise that all the pretzels in the families $p + q = 2, 4$ with unknotting number one are 2-bridge, however different technology is required to make any further progress.

5. The Slice Case

In this section, we tackle the slice case, namely $P(k, -k, 2m)$. Our general method is as follows: first, we pin down the value of $m = 1$, using the Alexander module (which we will define). We then use the correction term obstruction due to Ozsváth and Szabó to prove that the only such knot with unknotting number one is $P(3, -3, 2)$, up to reflection. Our ultimate goal is the following theorem.

Theorem 5.1. *Suppose K is a slice pretzel knot with $u(K) = 1$. Then $K = P(3, -3, 2)$, up to reflection.*

The determinant of $P(k, -k, 2m)$ is always k^2. Hence, in the Montesinos theorem, we can always take $D = k^2$, and so $n = \frac{k^2+1}{2}$.

5.1. **First Results.** Recall that these knots are slice, so we obtain no useful information from the Rasmussen invariant. In fact, we believe that direct application of Heegaard Floer homology is necessary to complete the problem; the computational difficulties this entails motivate us to make as much progress as possible beforehand with more elementary techniques. One such technique is the Alexander module.

Recall that we can construct the infinite cyclic cover X_∞ of a knot. This has a deck transformation group $\mathbb{Z}$, generated by some element t. Then $H_1(X_\infty;\mathbb{Z})$ is a $\mathbb{Z}[t,t^{-1}]$-module A, called the *Alexander module*, from which much topological information can be extracted. This is done via the r^{th} *elementary ideal*, denoted A_r, which is the ideal of $\mathbb{Z}[t,t^{-1}]$ spanned by the $(n-r+1)$-minors of any $n\times n$ matrix presentation for A.

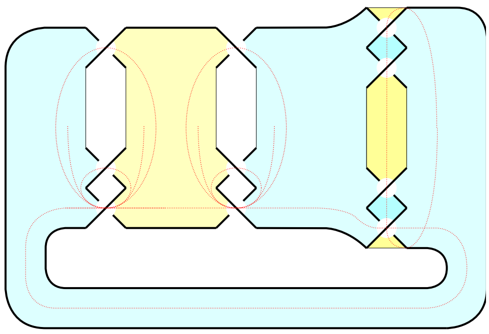
From Nakanishi [16], in the form cited in Lickorish [12], we know that the Alexander module can bound the unknotting number. For our purposes, we present the following definition-theorem (see Proposition 6.3 and Corollary 9.2 of Lickorish, [12]).

Theorem 5.2 (Unknotting via Alexander module)**.** *Suppose that V, an $n\times n$ matrix, is a Seifert matrix for K in S^3. Then the Alexander module of K is presented by the matrix $A = tV - V^t$. Moreover, if $\mathbb{Z}[t,t^{-1}]/A_r \neq 0$, it follows that $u(K)\geq r$.*

Using this, we present the following partial result for the slice pretzel case.

Lemma 5.3. *If $P(k,-k,2m)$, for $k\geq 3$, $m>0$, has unknotting number one, then $m=1$.*

Proof. We take the following Seifert surface for our pretzels, $P(k,-k,2m)$. The curves are indexed starting with the leftmost column of loops, smallest to largest, then the same for the next column. Finally, the big loop and the loop which pierces the "bridge". As regards orientations, we take the blue to be oriented out of the page, the yellow into the page.



From this, we find that a Seifert matrix for $P(k,-k,2m)$ has the form

$$V = \begin{pmatrix} X_k & 0 & 0 & 0 \\ 0 & -X_k & 0 & 0 \\ \mathbf{1} & -\mathbf{1} & 0 & 0 \\ 0 & 0 & 1 & m \end{pmatrix}$$

where X_k is the $(k-1)\times(k-1)$ lower triangular matrix of 1's, and $\mathbf{1}$ is a suitably sized rows of 1's.

From this, the Alexander module is presented by

$$A = \begin{pmatrix} M_k & 0 & -\mathbf{1}^t & 0 \\ 0 & -M_k & \mathbf{1}^t & 0 \\ \mathbf{t} & -\mathbf{t} & 0 & -1 \\ 0 & 0 & t & m(t-1) \end{pmatrix}$$

from which we can compute the $(2k-3)\times(2k-3)$ minors. Here, $M_k = tX_k - X_k^t$, and $\mathbf{t}$ is a row with all entries t.

We claim, that for $k\geq 3$, the second elementary ideal, A_2, generated by these minors (in $\mathbb{Z}[t,t^{-1}]$), is precisely given by

$$A_2 = \left\langle \sum_{i=0}^{k-1}(-1)^i t^{k-1-i}, m(t-1)\right\rangle$$

For the moment, we assume this, and call the first polynomial $\mathcal{P}_k(t)$. Then we show $A_2 = \mathbb{Z}[t,t^{-1}]$ if and only if $m=\pm 1$, since k is odd. Indeed, the quotient $\mathbb{Z}[t,t^{-1}]/\langle \mathcal{P}_k(t)\rangle$ is composed of all integral Laurent polynomials with the form

$$a_{k-2}t^{k-2} + a_{k-3}t^{k-3} + \cdots + a_1 t + a_0$$

together with their unit multiples (that is, multiples of t^n for n an integer). These are forced to be zero in A/A_2 if and only if they fall in the ideal $\langle m(t-1)\rangle$. In particular, we require all a_i to be divisible by m. This statement then implies $m = \pm 1$.

When $m = \pm 1$, observe that $\mathcal{P}_k(t)$ is in fact, for k odd,

$$\mathcal{P}_k(t) = t^{k-1} - t^{k-3}(t-1) - t^{k-5}(t-1) - \cdots - (t-1)$$

which means that in the quotient $\mathbb{Z}[t, t^{-1}]/\langle m(t-1)\rangle$, the polynomial is a unit, since $\mathcal{P}_k(t) \equiv t^{k-1}$, whence $\mathbb{Z}[t, t^{-1}]/A_2 = 0$. Hence there is no obstruction to unknotting number one, since the theorem guarantees $u(K) \geq 1$.

What remains then is to check our claim. As a first step, we can compute the determinant of M_k, which goes as follows. Here, for a row vector $\mathbf{v}$, we use the notation $\mathbf{v}^*$ to indicate a square matrix with each row $\mathbf{v}$.

$$\begin{aligned}
\det M_k &= \det \begin{pmatrix} t-1 & -1 & -1 & \dots & -1 \\ t & t-1 & -1 & \dots & -1 \\ t & t & t-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & t & \dots & t-1 \end{pmatrix} \\
&= t^{-1} \det \begin{pmatrix} t^2 & 0 & 0 & \dots & -1 \\ t & t-1 & -1 & \dots & -1 \\ t & t & t-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t & t & t & \dots & t-1 \end{pmatrix} \\
&= t \det M_{k-1} + (-1)^{k-1} t^{-1} \det \begin{pmatrix} \mathbf{t}^t & M_{k-2} \\ t & \mathbf{t} \end{pmatrix} \\
&= t \det M_{k-1} + (-1)^{k-1} \det \begin{pmatrix} 0 & M_{k-2} - \mathbf{t}^* \\ 1 & \mathbf{1} \end{pmatrix} \\
&= t \det M_{k-1} - \det(M_{k-2} - \mathbf{t}^*) \\
&= t \det M_{k-1} + (-1)^{k-1}
\end{aligned}$$

From this recurrence we can see that $\det M_k = \mathcal{P}_k(t)$. Now it is not hard to see that for any other minor, supposing that the $m(t-1)$ entry remains, we can expand down the final column or row. This yields two terms, one that contains $m(t-1)$ as a factor, and the other of which is the determinant of a diagonal matrix, one factor of which is either $\det(M_k)$ or $\det(-M_k)$, both of which are $\mathcal{P}_k(t)$ up to sign. The remaining case, when $m(t-1)$ is removed, is a calculation very much like that following this paragraph, and therefore has $\mathcal{P}_k(t)$ as a factor. What remains to be done in order to prove that A_2 is spanned by these two key polynomials is to ensure that they are both actually in the ideal. This is proved by the following two example minors.

First, we delete the first row and final column:

$$\begin{aligned}
\det A^{1,2k} &= \det \begin{pmatrix} \mathbf{t}^t & M_{k-1} & 0 & -\mathbf{1}^t \\ 0 & 0 & -M_k & \mathbf{1}^t \\ t & \mathbf{t} & -\mathbf{t} & 0 \\ 0 & 0 & 0 & t \end{pmatrix} \\
&= t \det \begin{pmatrix} \mathbf{t}^t & M_{k-1} & 0 \\ 0 & 0 & -M_k \\ t & \mathbf{t} & -\mathbf{t} \end{pmatrix} \\
&= -t^2 \det \begin{pmatrix} M_{k-1} - \mathbf{t}^* & \mathbf{t} \\ 0 & -M_k \end{pmatrix} \\
&= (-1)^k t^2 \det(M_{k-1} - \mathbf{t}^*) \det M_k \\
&= t^2 \mathcal{P}_k(t)
\end{aligned}$$

The last equality uses our previous calculation, and the fact that $M_{k-1} - \mathbf{t}^*$ is an upper-triangular matrix with all its $(k-2)$ diagonal entries being -1. Since t^2 is a unit, we know that $\mathcal{P}_k(t)$ is in A_2. We

now check that $m(t-1)$ is too, as evidenced by the following minor.

$$\begin{aligned}
\det A^{1,k+1} &= \det \begin{pmatrix} \mathbf{t}^t & M_{k-1} & 0 & -\mathbf{1}^t & 0 \\ 0 & 0 & \mathbf{1} & 1 & 0 \\ 0 & 0 & -M_{k-1} & \mathbf{1}^t & 0 \\ t & \mathbf{t} & -\mathbf{t} & 0 & -1 \\ 0 & 0 & 0 & t & m(t-1) \end{pmatrix} \\
&= m(t-1) \det \begin{pmatrix} \mathbf{t}^t & M_{k-1} & 0 & -\mathbf{1}^t \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & -M_{k-1} & \mathbf{1}^t \\ t & \mathbf{t} & -\mathbf{t} & 0 \end{pmatrix} \\
&= m(t-1) \det \begin{pmatrix} 0 & M_{k-1} - \mathbf{t}^* & \mathbf{t}^* & -\mathbf{1}^t \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & -M_{k-1} & \mathbf{1}^t \\ t & \mathbf{t} & -\mathbf{t} & 0 \end{pmatrix}
\end{aligned}$$

The last matrix determinant is then manipulated as

$$-t \det(M_{k-1} - \mathbf{t}^*) \det \begin{pmatrix} \mathbf{1} & 1 \\ -M_{k-1} & \mathbf{1}^t \end{pmatrix} = (-1)^{k-1} \det \begin{pmatrix} M_{k-1} & -\mathbf{1}^t \\ \mathbf{t} & t \end{pmatrix}$$

and this in turn is almost M_k. Thus, this determinant is

$$\det M_k - (t-1) \det M_{k-1} + t \det M_{k-1} = \det M_k + \det M_{k-1} = t^{k-1}$$

and we see then that, killing off unit factors, $m(t-1)$ is in A_2, at last completing our proof. □

Knowing this, we can now try to pin down the value of k. With this goal in mind, we use a healthy dose of Heegaard Floer homology. Before continuing, we remember that $m > 0$, since reflection leaves unknotting number unchanged, so we will assume henceforth that $m = 1$.

5.2. **Donaldson diagonalisation and $\Sigma(k, -k, 2)$.** We can mimic the work in Subsection 4.1, when the signature of K vanishes, using the Donaldson diagonalisation argument to pin down the sign of the unknotting crossing. Since this will be relevant in the next section on the Heegaard Floer homology obstruction, we present the result.

Lemma 5.4. *If $K = P(k, -k, 2)$ has unknotting number one, and $k \geq 3$, then it is undone by changing a negative crossing.*

Proof. Suppose that K is undone with a positive crossing. Then, $\overline{K}$ is undone by changing a negative crossing. Hence, in the signed Montesinos theorem (Theorem 4.2), $\Sigma(\overline{K}) = S^3_{-D/2}(\kappa)$, where κ is a knot in S^3 and $D = \det \overline{K} = \det K$, and so $-\Sigma(K)$ bounds a negative-definite 4-manifold with intersection form R_n.

The same plumbing we have used before is negative-definite, so we glue these two 4-manifolds together and there should be a diagonalising matrix appearing as

$$\left(\begin{array}{ccccccc|cc}
-1 & 1 & & & & & & & \\
 & -1 & 1 & & & & & & \\
 & & -1 & 1 & & & & & \\
 & & & \ddots & \ddots & & & & \\
\vdots & \vdots & \vdots & & -1 & 1 & & & \\
 & & & & & -1 & 1 & & \\
a & a & a & \dots & a & b & b & c & c \\
\hline
d & d & d & \dots & d & d & d & 1 & \\
 & & & & & & & 1 & -1
\end{array}\right)$$

Denote the rows by v_i, with a total of $k+4$ rows. Then $v_k \cdot v_{k+2} = -1$, so $b = a-1$. Then $v_{k+2} \cdot v_{k+2} = k$ implies

$$ka^2 + 2(a-1)^2 + 2c^2 = k \tag{3}$$

whence evidently we must have $a = 0, 1$ (else the LHS is too big). We split the cases:

(1) If $a = 0$, then from (3) we have $2c^2 + 2 = k$. This is nonsense for parity reasons.
(2) If $a = 1$, then $c = 0$ (from (3)). Then $v_{k+2} \cdot v_{k+3} = 0$ tells us $kd = 0$, whence $d = 0$. The fact that $v_{k+3} \cdot v_{k+3} = n$ yields up $n = 1$, so $k^2 = 1$, contradicting $k \geq 3$.

This completes the proof. □

6. The Heegaard Floer Homology of $\Sigma(k, -k, 2)$

We compute the Heegaard Floer homology of $\Sigma(k, -k, 2)$, with an end to deducing which of the corresponding pretzels have unknotting number one. We first refer to Ozsváth and Szabó, [20], who have presented a combinatorial algorithm for determining the Heegaard Floer homology of plumbed three-manifolds, which is clearly the case here (we have small Seifert fibred spaces). Throughout this and the following sections, we implicitly only care about $k \geq 3$, since $k = 1$ yields the unknot.

Since we employ the algorithm described in Section 3 of [20], we describe it briefly now. What is more, to be clearer, we elaborate on our discussion of plumbed manifolds from the previous section. Recall that a 4-manifold $X = X(G)$ with boundary $Y = Y(G)$ can be constructed from a graph G, then suppose that $\mathrm{Char}(G)$ is the set of all characteristic vectors for the plumbing, namely those such that

$$\langle K, v \rangle \equiv [v] \cdot [v] \mod 2$$

Now, it is well known that the Spin^c-structures on X correspond precisely with $\mathrm{Char}(G)$; by mapping $H_2(X)$ to $H^2(X, \partial X)$ via Poincaré duality, we see that the image in $H^2(X)$ is spanned by $\mathrm{PD}[v]$. Moreover, the Spin^c-structures on ∂X correspond to the $2H^2(X, \partial X)$-orbits of $\mathrm{Char}(G)$.

The algorithm that Ozsváth and Szabó give in [20] states that, assuming there is at most one overweight vertex in the graph, where a vertex is *overweight* if

$$w(v) > -d(v)$$

where $d(v)$ is the degree of v, the correction term of Y for $\mathfrak{t}$ can be computed by looking at all the characteristic vectors in Spin^c-structure $\mathfrak{t}$, and putting

$$d(Y, \mathfrak{t}) = \frac{1}{4}\left(\max_{K \in \mathrm{Char}_{\mathfrak{t}}(G)} K^2 + |G| \right)$$

Note that the square here is taken with the metric Q^{-1}, namely the dual metric. Incidentally, this shows that such 4-manifolds are sharp.

A $K \in \mathrm{Char}(G)$ that satisfies,

$$w(v) + 2 \leq \langle K, v \rangle \leq -w(v)$$

begins a path $(K = K_0, K_1, \ldots, K_n)$ such that for $i \leq n$

$$|\langle K_i, v \rangle| \leq -w(v)$$

for vertices v, and for some choice of vertex v_{i+1} such that

$$\langle K_i, v_{i+1} \rangle = -w(v_{i+1})$$

we set $K_{i+1} = K_i + 2\,\mathrm{PD}[v_{i+1}]$ (which we will refer to as *pushing down* v_{i+1}). The algorithm terminates in one of two ways: either, for any vertex v,

$$w(v) \leq \langle K_n, v \rangle \leq -w(v) - 2$$

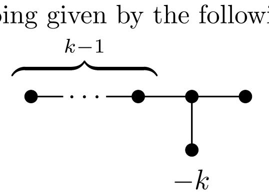
in which case we say the path was *maximising*, or, for some vertex v,

$$\langle K_n, v \rangle > -w(v)$$

in which case the path was *non-maximising.*

Ozsváth and Szabó moreover show (Proposition 3.2 of [21]) that the maximisers required for computing correction terms can be taken from a set of characteristic vectors $\mathrm{Char}^*(G)$ with the property that they initiate a maximising path. We now apply this idea. Indeed, the first part of Theorem 2.2 in [22] tells us immediately that $\Sigma(k, -k, 2)$ is an L-space, and thus we must have at least one maximising path per Spin^c-structure.

We note that $\Sigma(k, -k, 2)$ has a plumbing given by the following diagram.



Unlabelled vertices have weight -2. This is indeed negative definite, and has only one overweight vertex (the central node). These vertices are labelled v_1 to v_{k+1}, from left to right, with v_k being the central (degree-3) node. The final leaf is v_{k+2}. Under this choice of basis, our intersection form Q has matrix

$$Q = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 1 & 1 \\ 0 & 0 & 0 & \dots & 1 & -2 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & -k \end{pmatrix}$$

We denote the characteristic vectors by their values on each vertex.

Proposition 6.1. *The following characteristic vectors initiate maximising paths, in the sense above.*

(1) $(0,0,\dots,0,2,0,\dots,0,j)$, *where the* 2 *is in the* i^{th} *place, and* $j \in \mathbb{Z}$ *is odd and* $2-k \leq j \leq k-4$*;*
(2) $(2,0,\dots,0,0,k-2)$ *and* $(0,\dots,0,2,k-2)$*; and*
(3) $(0,\dots,0,j)$ *where* j *is an odd integer satisfying* $2-k \leq j \leq k$*.*

There are no other vectors that initiate full paths. Observe that there are $(k+1)(k-2) = k^2-k-2$ *such vectors of the first kind, 2 of the second, and* k *of the last, giving the full* k^2*.*

Proof. We show these case by case. First, though, we introduce a trick we call *migration*. Suppose that we have some string $(0,2,0,\dots,0,-2)$, which avoids the last three slots, then apply

$$(0,2,0,0,\dots,0,-2) \to (2,-2,2,0,\dots,0,-2) \to (2,0,-2,2,\dots,0,-2)$$

At this point, we push down the 2's to the right, and end with $(2,0,\dots,-2,0)$. We have shifted the pattern one to the left. An analogous move applies migrating $(-2,0,\dots,0,2)$ to the right. Now to apply this:

(1) For $i \leq k-1$, we take the vertex labelled 2, and blow it down. We then successively push down the 2's to the left and right, until we eventually have a vector of the form $(-2,0,\dots,0,2,0,\dots,0,j)$. We migrate the string $(-2,\dots,2)$ to the right, giving eventually

$$\begin{aligned}(0,\dots,-2,\dots,0,0,2,0,j) &\to (0,\dots,-2,\dots,0,2,-2,2,j+2) \\ &\to (0,\dots,-2,\dots,0,2,0,-2,j+2) \\ &\to (0,\dots,-2,\dots,2,-2,2,-2,j+2) \\ &\to (0,\dots,-2,\dots,2,0,-2,0,j+4)\end{aligned}$$

We can then migrate the $(2,0,-2)$ to the left and thus eventually eliminate all 2's. We are done unless $j+4=k$. In this case, from the end of the above path we migrate the $(-2,\dots,2)$ right until we have $(0,\dots,-2,\dots,0,0,k)$, and then pushing down the k we find ourselves with $(0,\dots,-2,\dots,2,0,-k)$. This has already been covered (formally) above, at the start of the path above.

The other options are, first,

$$\begin{aligned}(0,\dots,2,0,j) &\to (0,\dots,2,-2,2,j+2) \\ &\to (0,\dots,2,0,-2,j+2)\end{aligned}$$

at this point, we mimic the above, center-aligned path from the second line (the only difference being that we have no -2 sitting to the left); when we migrate the $(2,0,-2)$ left, we can then blow down the 2 to get $(-2,2,-2)$, and one final push-down eliminates all 2's. We are thus done unless $j+4=k$. If that is the case, we blow this vertex down and find ourselves in a case (formally) just covered. Finally,

$$\begin{aligned}(0,\dots,0,0,2,j) &\to (0,\dots,0,2,-2,j) \\ &\to (0,\dots,2,-2,0,j+2)\end{aligned}$$

and push down the 2 to the left.

(2) For the 2 in the far left, we push the 2 down to the right, until we have

$$\begin{aligned}(0,\dots,-2,2,0,k-2) &\to (0,\dots,0,-2,2,k)\\ &\to (0,\dots,0,0,-2,k)\\ &\to (0,\dots,0,2,-2,-k)\\ &\to (0,\dots,2,-2,0,2-k)\end{aligned}$$

and push the 2 down to the left. For the 2 in the far right,

$$\begin{aligned}(0,\dots,0,0,0,2,k-2) &\to (0,\dots,0,0,2,-2,k-2)\\ &\to (0,\dots,0,2,-2,0,k)\\ &\to (0,\dots,2,-2,0,-2,k)\\ &\to (0,\dots,2,-2,2,-2,-k)\\ &\to (0,\dots,2,0,-2,0,2-k)\end{aligned}$$

and we then push down to migrate the $(2,0,-2)$ to the left, eventually eliminating all 2's.

(3) Unless $j = k$, we are done. In this case, we push it done to obtain the vector $(0,\dots,0,2,0,-k)$, which is formally the same as a vector in the first family (even though j lies outside the range, the only relevant fact there was that $j \le k-4$).

This gives all the relevant characteristic vectors. It is not hard to check that they are all the possibilities, by showing, for example, that vectors with two 2's initiate non-maximising paths, and similarly with 2 and k in the same vector (eventually, after pushing up and down, we end up with an entry that's too big). □

The significance here is that since $\Sigma(k,-k,2)$ is an L-space, each element above is a maximiser, and therefore they are a complete set of representatives for the Spin^c-structures on $\Sigma(p,-p,2)$. By calculating their lengths we can compute the corresponding correction terms. Moreover, $\mathrm{Char}^*(G)$ has a cyclic group structure, which we now describe.

First, we need an equivalence relation $\sim$ on $H^2(X)$. We say $v \sim v'$ if $(v-v')Q^{-1} \in \mathbb{Z}^{k+2}$. It can be shown that the equivalence classes are in fact represented by $\mathrm{Char}^*(G)$, since the equivalence classes also represent $\mathrm{Spin}^c(\partial X)$.

The cyclic group structure on $\mathrm{Char}(G)$ then appears as an operation $\widehat{+}$ defined

$$K_1 \widehat{+} K_2 := K$$

for $K_1, K_2 \in \mathrm{Char}^*(G)$ where K is the unique element in $\mathrm{Char}^*(G)$ such that $K_1 + K_2 \sim K$.

The labelling of the Spin^c-structures is essentially a combinatorial problem. We give the maximisers $\mathrm{Char}^*(G)$ the following names:

$$\begin{aligned}
K^1_{ij} &:= (0,\dots,2_{(i)},\dots,0,j) && j \text{ odd},\ 2-k \le j \le k-4\\
K^2_1 &:= (2,0,\dots,k-2) && K^2_2 := (0,\dots,0,2,k-2)\\
K^3_j &:= (0,\dots,0,j) && j \text{ odd},\ 2-k \le j \le k
\end{aligned}$$

It will sometimes be useful to allow K^1_{ij} to define the same type of vector as above, but with a value for j outside the range and parity specified. In this case we emphasise that it does not represent a maximiser useful for calculating the corresponding correction term. It is also notationally useful to set

$$K^1_{0,j} := K^3_j$$

We now have the task of calculating to which Spin^c structure each of the $\mathrm{Char}^*(G)$ elements corresponds. Once we have done this, we have the correction terms, since then,

$$d(\Sigma(k,-k,2),i) = \frac{K_i Q^{-1} K_i^t + (k+2)}{4}$$

where $K_i \in \mathrm{Char}^*(G)$ is the characteristic vector corresponding to the Spin^c-structure $\mathfrak{t}_i$, where we have $\frac{1}{2}c_1(\mathfrak{t}_i) = i \in H_1(\Sigma(k,-k,2)) = \mathbb{Z}/(k^2)$. This requires knowledge of Q^{-1}, which is a surprisingly tractable calculation, as follows.

Since Q is symmetric, so is Q^{-1}, and thus applying Cramer's rule we can get away with computing only half the minors of Q. We know that the A-series root system has determinants $\det A_n = (-1)^n(n+1)$; moreover, we also know that the plumbings can be slam-dunked to obtain star graphs with at most 4

vertices, the centre of weight 0, yielding a determinant of $w_1w_2 + w_2w_3 + w_1w_3$, the w_i being the weights of these leaves. These two computations allow us to see very fast that the inverse is $Q^{-1} = \frac{1}{k^2}(c_{ij})$, where

$$c_{ij} = \begin{cases} -i(k^2 - jk + 2j) & i \le j \le k-1 \\ -2jk & i = k, j \le k \\ -jk & i = k+1, j \le k \\ -k^2 & i = j = k+1 \\ -2j & i = k+2, j \le k \\ -k & i = k+2, j = k+1 \\ -(k+2) & i = j = k+2 \\ c_{ji} & \text{all } i, j \end{cases}$$

This in turn permits an explicit calculus of the correction terms. Let $D : \mathrm{Char}^*(G) \longrightarrow \mathbb{Q}$ be the squaring map determined by

$$D(K) = 4d(\Sigma(k, -k, 2), \phi^{-1}(K)) - (k+2) = KQ^{-1}K^t$$

Then:

$$\begin{aligned} D(K^1_{ij}) &= -\tfrac{1}{k^2}(4i(k^2 - ik + 2i) + (k+2)j^2 + 8ij) \\ D(K^1_{kj}) &= -\tfrac{1}{k^2}(8k^2 + (k+2)j^2 + 8kj) \\ D(K^1_{k+1,j}) &= -\tfrac{1}{k^2}(4k^2 + (k+2)j^2 + 4kj) \\ D(K^2_1) &= -(k+2) \\ D(K^2_2) &= -\tfrac{1}{k^2}(k^3 + 6k^2 - 12k + 8) \\ D(K^3_j) &= -\tfrac{1}{k^2}(k+2)j^2 \end{aligned}$$

where $i \le k-1$, and j satisfying the appropriate conditions. We observe that $D(K^1_{0,j}) = D(K^3_j)$, so our notational trick is consistent. The computation of $d(Y, K)$ is then trivial.

6.1. **The Lens Spaces $L(k^2, 2)$.** We need to repeat this procedure for the corresponding lens space, $L(k^2, 2)$, which has a plumbing given by the tree on two vertices, labelled -2 and $-n$ (where $k^2 = 2n-1$). It therefore has intersection form given by

$$R = \begin{pmatrix} -n & 1 \\ 1 & -2 \end{pmatrix}$$

and in this case the inverse is trivially

$$R^{-1} = -\frac{1}{k^2}\begin{pmatrix} 2 & 1 \\ 1 & n \end{pmatrix}$$

What remains is to establish the labeling of the Spin^c-structures on $L(k^2, 2)$ as they are required for the symmetry theorem, and to compute the correction terms. The first is in fact already done by Ozsváth and Szabó in [20]. We quote the result now.

Lemma 6.2 (Ozsváth and Szabó)**.** *The lens space $L(2n-1, 2)$ has characteristic vectors given by the map $\psi : Spin^c(L(2n-1, 2)) \longrightarrow Char^*(H)$, where H is the obvious plumbing of the lens space, and*

$$\psi(i) = \begin{cases} (2i-1, 2) & 0 \le i \le s \\ (2i-4s-1, 0) & s+1 \le i \le 3s+1 \\ (2i-8s-3, 2) & 3s+2 \le i \le 4s \end{cases} \tag{4}$$

and $n = 2s+1$.

So we compute the correction terms, finding:

$$d(L(k^2, 2), i) = \frac{-\psi(i)\begin{pmatrix} 2 & 1 \\ 1 & n \end{pmatrix}\psi(i)^t + 2k^2}{4k^2}$$

or, explicitly,

$$d(L(k^2,2),i) = \begin{cases} -\frac{1}{k^2}(2i^2) & 0 \le i \le s \\ -\frac{1}{2k^2}((2i-k^2)^2 - k^2) & s+1 \le i \le 3s+1 \\ -\frac{1}{k^2}(2(k^2-i)^2) & 3s+2 \le i \le 4s \end{cases}$$

We may denote this as $d(\cdot, i)$ or $d(\cdot, \psi(i))$, with obvious identifications, according to whichever notation makes our purposes clearer later.

6.2. **Obstruction to Unknotting Number One.** We are now finally ready to apply a theorem in Ozsváth and Szabó [21], namely the symmetry property mentioned in their Theorem 1.1. More accurately, we are using Theorem 4.1, which suffices for us, and does not require any alternating knots (that said, even Theorem 1.1 only relies on certain assumptions on the Heegaard Floer homology of the double branched covers).

Recall their Theorem 4.1 (henceforth called the *unknotting theorem*) which reads as follows.

Theorem 6.3 (Ozsváth-Szabo unknotting theorem, applied to our pretzels)**.** *If $d(\Sigma,0) = d(L,0)$, and Σ is an L-space, then for some isomorphism $\widetilde{\phi} : \mathbb{Z}/(k^2) \longrightarrow Char^*(G)$, we have*

$$d(\Sigma, \widetilde{\phi}(i)) - d(\Sigma, \widetilde{\phi}(2s-i)) = -\epsilon\,[d(L,\psi(i)) - d(L,\psi(2s-i))] \tag{5}$$

where $\Sigma := \Sigma(k,-k,2)$ and $L := L(k^2,2)$, $\epsilon \in \{\pm 1\}$, and i runs from 0 to s.

The sign, $\epsilon \in \{\pm 1\}$, is the sign of the crossing at which the unknotting takes places, and comes from the ambiguity in the Montesinos theorem regarding whether $\Sigma(k,-k,2)$ bounds a positive or negative definite manifold. Our aim for this section then is to prove that $\Sigma(k,-k,2)$ satisfies the requirements of the theorem.

Our first stop on this track, then, is to find any old isomorphism

$$\phi : \mathrm{Spin}^c(\Sigma) = \mathbb{Z}/(k^2) \longrightarrow \mathrm{Char}^*(G)$$

which we will later modify into the specific one $\widetilde{\phi}$ discussed above. To specify any isomorphism ϕ entirely, we only need $\phi(1)$. Thus, it behooves us to locate the zero and a unit in $\mathrm{Char}^*(G)$, neither of which are immediately obvious. The zero is the unique element of $\mathrm{Char}^*(G)$ such that $KQ^{-1} \in \mathbb{Z}^{k+2}$, since we want $K \widehat{+} K = K$.

Observe, if $v_i = (K_1^2 Q^{-1})_i$, then for $i \le s-1$,

$$\begin{aligned} v_i &= \tfrac{1}{k^2}(2c_{i,1} + (k-2)c_{i,k+2}) = -2 \\ v_k &= \tfrac{1}{k^2}(2c_{k,1} + (k-2)c_{k,k+2}) = -2 \\ v_{k+1} &= \tfrac{1}{k^2}(2c_{k+1,1} + (k-2)c_{k+1,k+2}) = -1 \\ v_{k+2} &= \tfrac{1}{k^2}(2c_{k+2,1} + (k-2)c_{k+2,k+2}) = -1 \end{aligned}$$

which tells us that K_1^2 is the zero element in $\mathrm{Char}^*(G)$.

Now, to find a unit, K; that is, m copies of K under $\widehat{+}$ should only yield the zero K_1^2 if and only if m is a multiple of k^2. Equivalently, we seek a K so that

$$(mK - K_1^2)Q^{-1} \in \mathbb{Z}^{k+2}$$

if and only if m is a multiple of k^2. Suppose that $K = K_{1,-1}^1$, from which we have

$$mK - K_1^2 = (2(m-1), 0, \ldots, 0, -m-(k-2))$$

requiring us to conclude (computing as above),

$$[(mK - K_1^2)Q^{-1}]_{k+2} = \tfrac{1}{k^2}(k^2 - m(k-2)) \in \mathbb{Z}$$

It follows that $k^2 | m$, since k^2 and $k-2$ are coprime, k being odd. Thus $K_{1,-1}^1$ is a unit.

This gives us an isomorphism ϕ, by the defining equations:

$$\phi(0) = K_1^2 \qquad\qquad \phi(1) = K_{1,-1}^1$$

We are therefore ready to prove the required lemma. In what follows, we write $D(i) := D(\phi(i))$ where $i \in \mathbb{Z}/(k^2)$.

Lemma 6.4. *$P(k,-k,2)$ satisfies the conditions of the Ozsváth-Szabó unknotting theorem. More explicitly, $d(\Sigma,0) = d(L,0)$, and $\Sigma(k,-k,2)$ is an L-space.*

Proof. This is a straightforward computation. We already know $d(L, 0) = 0$ from (4), and from the above discussion $K = K_1^2$ is the zero-element (which is preserved under any isomorphism). Computing its correction term,

$$d(\Sigma, 0) = \frac{D(0) + |G|}{4} = \frac{(K_1^2)^t Q^{-1} K_1^2 + (k+2)}{4} = 0$$

For what remains, we have already discussed why $\Sigma(k, -k, 2)$ is an L-space. $\square$

6.3. **Modifying the Unknotting Theorem.** The unknotting theorem is difficult to compute with directly, since passing between different isomorphisms ϕ is not easily done. However, we can sacrifice equality for congruence modulo $\mathbb{Z}$, to narrow the search for our $\widetilde{\phi}$.

First, we attack the RHS of (5). Using Ozsváth and Szabó's calculation for lens spaces, we can compute the RHS:

$$f_L(k, i) := d(L, \phi(i)) - d(L, \phi(2s - i)) = -\tfrac{1}{2k^2}\left(k^2 - 4i - 1\right)$$

Now, to treat the LHS. Recall we have the isomorphism

$$\tfrac{1}{2}c_1 : \mathrm{Spin}^c(\Sigma(k, -k, 2)) \longrightarrow \mathbb{Z}/(k^2)$$

so then we can define the following map

$$\widetilde{D} : \mathrm{Isom}\left(\mathbb{Z}/(k^2), \mathrm{Char}^*(G)\right) \times \mathbb{Z}/(k^2) \times \mathbb{Z}/(k^2) \longrightarrow \mathbb{Q}$$

setting

$$\widetilde{D}(\phi, i, j) := \phi(i) Q^{-1} \phi(j)^t$$

and then we observe that the relevant expression for the unknotting theorem is

$$\tfrac{1}{4}\left(\widetilde{D}(\widetilde{\phi}, i, i) - \widetilde{D}(\widetilde{\phi}, 2s - i, 2s - i))\right)$$

once we have ascertained the specific isomorphism $\widetilde{\phi}$. Suppose then that we have already fixed a particular $\phi \in \mathrm{Isom}\left(\mathbb{Z}/(k^2), \mathrm{Char}^*(G)\right)$, then $\widetilde{\phi} = \varphi \circ \phi$ where $\varphi \in \mathrm{Aut}(\mathrm{Char}^*(G))$. We know from the group structure discussed on $\mathrm{Char}^*(G)$ that φ is a combination of multiplication in $H^2(X)$ by some $\ell \in \mathbb{Z}$, coprime to k^2, followed by the projection $H^2(X) \longrightarrow \mathrm{Char}^*(G)$ via $\sim$.

This all looks to be complicated, yet simplifies quickly on defining a map $D(\cdot, \cdot) = \widetilde{D}(\phi, \cdot, \cdot)$. We set

$$f_\Sigma(k, i) := -D(2s, 2s) + 2D(i, 2s)$$

Using this, we can construct the following weaker version of the unknotting theorem, but one which is easier to compute.

Theorem 6.5 (Symmetry theorem)**.** *With notation as above, for $P(k, -k, 2)$ to have unknotting number one, we require*

$$\ell^2 f_\Sigma(k, i) \equiv -4\epsilon f_L(k, i) \mod \mathbb{Z} \tag{6}$$

for some ℓ coprime to k^2, and $i \leq s$.

Proof. Recall that $\widetilde{\phi} = \varphi \circ \phi$, and φ is multiplication by ℓ then projection to $\mathrm{Char}^*(G)$. Thus, $\varphi \circ \phi(j)$ is equivalent to $\ell\phi(j)$ under the equivalence relation $\sim$ defined on $H^2(X)$. Consequently, we find that $[\varphi \circ \phi(i)]Q^{-1} = [\ell\phi(i)]Q^{-1} + v_i$, and similarly $Q^{-1}[\varphi \circ \phi(j)]^t = Q^{-1}[\ell\phi(j)]^t + v_j^t$ where $v_i, v_j \in \mathbb{Z}^{p+2}$. Then:

$$\begin{aligned}
\widetilde{D}(\widetilde{\phi}, i, j) &= \widetilde{\phi}(i)^t Q^{-1} \widetilde{\phi}(j) \\
&= [\varphi \circ \phi(i)] Q^{-1} [\varphi \circ \phi(j)]^t \\
&= [\ell\phi(i)] Q^{-1} [\ell\phi(j)]^t + v_i \cdot [\varphi \circ \phi(j)]^t + [\ell\phi(i)] \cdot v_j^t \\
&\equiv \ell^2 D(i, j) \mod \mathbb{Z}
\end{aligned}$$

We observe then that, by similar reasoning,

$$D(i, i) - D(2s - i, 2s - i) \equiv f_\Sigma(k, i) \mod \mathbb{Z}$$

and then the rest is just substitution into the unknotting theorem of Ozsváth and Szabó. $\square$

6.4. **Spin^c-Structures.** The goal in what follows is to cut down the automorphisms of $\mathbb{Z}/(k^2)$ which satisfy the symmetry theorem, then, under the assumption k prime, show that there is a unique solution. We show this fails the actual exact correction terms test (with equality, rather than congruence).

Proposition 6.6. *Suppose $P(k,-k,2)$ has unknotting number one. Then there exists an ℓ coprime to k^2 such that*

$$\ell^2(3k-2) \equiv -8 \mod k^2$$

for some choice of $\epsilon = \pm 1$. Equivalently,

$$\ell^2 \equiv 6k+4 \mod k^2 \tag{7}$$

Proof. We begin by computing

$$f_L(k,0) = -\frac{1}{2k^2}(k^2-1)$$

It is also a routine matter of calculation to find

$$\phi(2s) = \begin{cases} K^1_{k+1,-\frac{1}{2}(k+1)} & k \equiv 1 \mod 4 \\ K^3_{\frac{1}{2}(k-1)} & k \equiv 3 \mod 4 \end{cases}$$

from which we deduce that

$$f_\Sigma(k,0) = \begin{cases} \frac{1}{4p^2}(5k^3-3k+2) & k \equiv 1 \mod 4 \\ \frac{1}{4p^2}(-3k^3+4k^2-3k+2) & k \equiv 3 \mod 4 \end{cases}$$

Now, on substituting into the theorem, (6), we find that we must have

$$\begin{aligned} \ell^2(5k^3-3k+2) &\equiv 8\epsilon(k^2-1) \mod k^2 & k \equiv 1 \mod 4 \\ \ell^2(-3k^3+4k^2-3k+2) &\equiv 8\epsilon(k^2-1) \mod k^2 & k \equiv 3 \mod 4 \end{aligned}$$

From Proposition 5.4, we can substitute $\epsilon = -1$, and then both reduce to the required result. The equivalent statement (7) is a simple rearrangement (to make ℓ^2 the subject). □

The proof that $u(K) > 1$ for $k \geq 5$ then follows the following line of reasoning. We show that for such ℓ, we cannot satisfy (5) for $i = 0$ and r simultaneously, where r is the residue of ℓ modulo k. This requires us to do the following:

(1) Pinpoint the values of $\phi(2s\ell)$, $\phi(r\ell)$, $\phi(2s\ell - r\ell)$, and thence compute their squares;
(2) Compute the differences $d(\Sigma, i\ell) - d(\Sigma, 2s\ell - i\ell) - d(L,i) + d(L, 2s-i)$ for $i = 0, r$;
(3) Obtain a good reason why these two differences cannot be zero simultaneously for $k \geq 5$.

For the reader who does not like results plucked out of thin air, we can provide some comments on the combinatorics involved in the group structure on $\mathrm{Char}^*(G)$. The following formulae are the tools used to compute the values of ϕ called for in the first step above.

Lemma 6.7 (Exchange formulae)**.** *We have the following exchange formulae:*

$$\textbf{\textit{(A):}}\ K^3_{J+kB} \sim K^1_{-B,J+2B} \qquad \textbf{\textit{(B):}}\ K^1_{I,J} \sim K^1_{I+1,J+k-2} \qquad \textbf{\textit{(C):}}\ K^1_{I,J} \sim K^1_{I,J+k^2}$$

where the $\sim$ indicates that the vectors in $H^2(X)$ represent the same Spin^c-structure on $\partial X = \Sigma(K)$.

Proof. This is an easy calculation: simply verify that $(K-K')Q^{-1} \in \mathbb{Z}^{k+2}$, for the above K, K'. Notice, in fact, that (A) is actually a consequence of (B), iterated $-k$ times. However, in this particular form it is particularly useful, and thus we have stated it as is. □

If k is a prime power, then there is an essentially unique choice of ℓ (the only ambiguity being sign). The situation becomes much more complicated if k has several different prime factors; to deal with this complexity, we introduce some auxiliary notation.

Proposition 6.8. *Let $\ell = ak + r$, where $0 \leq a < k$ and $2 \leq a < k$. Then we can choose r even and set $r^2 = Ak + 4$, where*

$$A + 2ar \equiv 6 \mod k \tag{8}$$

There is also the inequality $0 \leq r - A < \frac{k}{4} + 1$.

Proof. Since $\pm\ell$ have the same effect on the correction terms, and k is odd, one of $\pm\ell$ will have even r and we make this choice. Notice that as ℓ is coprime to k, we cannot have $r=0$.

From our earlier congruence, (7), $\ell^2 \equiv 6k+4 \mod k^2$, but also $\ell^2 \equiv 2ark + r^2 \mod k^2$, and substituting gives the desired congruence (8).

For the inequality, we have $r - A = r - \frac{r^2-4}{k}$, and by considering this quadratic in the range from 0 to k, we find it is always positive, maximises when $r = \frac{k}{2}$, and has maximum $\frac{k}{4} + \frac{4}{k}$. Since $r - A$ is an integer, and as $k \geq 5$, the upper bound follows. $\square$

As remarked, there is an essentially unique choice of ℓ for k prime power. We give it explicitly below, by way of example.

Proposition 6.9. *In the case that k is a prime power, then $r = 2$, $A = 0$, and $a = \frac{k+3}{2}$.*

Proof. This is a direct calculation using (7), and the observation that when k is prime power, square roots modulo k are unique up to sign. $\square$

We now divide into two situations. The condition that r is even implies nothing about a, and since the parity of a becomes a subject in what follows, we divide along this line. Step One in our recipe calls for computation of various ϕ values. One of these, it turns out, is independent of a.

Proposition 6.10. *For $k \geq 5$ and a even, we have the mercifully simple equality*

$$\phi(r\ell) = -K^1_{\frac{A}{2},k-4-A}$$

Proof. By direct verification. Check that $(-r\ell K^1_{1,-1} - K^1_{\frac{A}{2},k-4-A})Q^{-1} \in \mathbb{Z}^{k+2}$, which is easy to do. $\square$

One final notational remark. Strictly speaking, we want to compute $d(\Sigma, i)$, but as there is the conjugation symmetry $d(Y,i) = d(Y,-i)$, sometimes we will in fact compute $\phi(-i)$ instead of $\phi(i)$. In these instances, we may write $\phi(i) = -K$, where it is understood that the minus sign comes from the $\widehat{+}$ group structure. We have done this in the proposition just completed, and do so in what follows, but this changes nothing for our final computations.

6.5. **The case** $a \in 2\mathbb{Z}$**.** Following Step One, we now compute $\phi(2s\ell), \phi(2s\ell - r\ell)$. This is done in the following two propositions.

Proposition 6.11 ($r \equiv 2 \mod 4$)**.** *If $r \equiv 2 \mod 4$ and a is even, set $B := 1 + \frac{r}{2} - \frac{a}{2} - \frac{A}{2} \in (-\frac{k}{2}, \frac{k}{4})$ and $J := -\frac{r}{2} - 4 < 0$. These give*

$$\phi(2s\ell - r\ell) = \begin{cases} K^1_{-B,J+2B} & B \leq 0, J+2B > -k \\ K^1_{2-B,J+2B+2k-4} & B \leq 0, J+2B \leq -k \\ -K^1_{B,-J-2B} & B \geq 0 \end{cases}$$

and also

$$\phi(2s\ell) = \begin{cases} K^1_{\frac{a-r}{2},\frac{r}{2}-a} & \text{if } a \geq r \\ -K^1_{\frac{r-a}{2},a-\frac{r}{2}} & \text{if } a \leq r \end{cases}$$

Proposition 6.12 ($r \equiv 0 \mod 4$)**.** *If, on the other hand, $r \equiv 0 \mod 4$, set $B := \frac{r}{2} - \frac{a}{2} - \frac{A}{2} \in (-\frac{k}{2}, \frac{k}{4})$ and $J := k - \frac{r}{2} - 4 > 0$, giving*

$$\phi(2s\ell - r\ell) = \begin{cases} K^1_{-B,J+2B} & B \leq 0 \\ -K^1_{B,-J-2B} & B \geq 0, J+2B < k \\ -K^1_{B+2,-J-2B+2k-4} & B \geq 0, J+2B \geq k \end{cases}$$

and also

$$\phi(2s\ell) = \begin{cases} K^1_{\frac{a-r+2}{2},\frac{r}{2}-a+k-2} & a \geq r-2 \\ -K^1_{\frac{r-a-2}{2},a-\frac{r}{2}-k+2} & a \leq r-2 \end{cases}$$

Proof (of both propositions). This is a straightforward verification. To perform it, one need only check that $(mK^1_{1,-1} - K)Q^{-1} \in \mathbb{Z}^{k+2}$ for the right choices of m and K from the above. In doing so, one must use the congruence (8) to guarantee the result. The numerous cases occur to fit the various restraints imposed on i, j in $K^1_{i,j}$; the fact that $\frac{r}{2}$ is odd makes a difference is because of the fact that j must be odd.

For the interested reader, these calculations were performed originally by assuming that K had the form K^3_j, and then applying the three exchange formulae above until the subscripts fitted their required conditions. ☐

This completes Step One. The next step is to use the formula for $D(K^1_{i,j})$ to compute the correction terms and ultimately the differences

$$Z(i) := d(\Sigma, i\ell) - d(\Sigma, 2s\ell - i\ell) - d(L, i) + d(L, 2s - i)$$

by observing that

$$Z(i) = \tfrac{1}{4}D(\phi(i\ell)) - \tfrac{1}{4}D(\phi(2s\ell - i\ell)) + \tfrac{2i^2}{k^2} - \tfrac{1}{2k^2}((2i+1)^2 - k^2)$$

On doing so, we find that the expressions that result all have a denominator of $16k^2$, so what we present in the table below is $16k^2 Z(i)$ (check the header of the column). The "case" label will become relevant later. Without further ado we present the table first for $i = r$:

Case	$r \mod 4$	Conditions	$16k^2 Z(r)$
A	2	$B \leq 0$ $J + 2B > -k$	$(4kr + 8k^2)A + (2-3k)r^2 + ((4a-24)k - 8k^2)r$ $-4k^3 + (8a+16)k^2 + 32ak - 8$
B	2	$B \leq 0$ $J + 2B \leq -k$	$(4kr - 8k^2)A + (2-3k)r^2 + (4a-24)kr$ $+12k^3 + (-8a-16)k^2 + 32ak - 8$
C	2	$B \geq 0$	$(4kr - 8k^2)A + (2-3k)r^2 + ((4a-24)k + 8k^2)r$ $-4k^3 + (-8a+48)k^2 + 32ak - 8$
D	0	$B \leq 0$	$4krA + (2-3k)r^2 + ((4a-24)k - 4k^2)r$ $+8k^2 + 32ak - 8$
E	0	$B \geq 0$ $J + 2B < k$	$(4kr - 16k^2)A + (2-3k)r^2 + ((4a-24)k + 12k^2)r$ $+(-16a+8)k^2 + 32ak - 8$
F	0	$B \geq 0$ $J + 2B \geq k$	$4krA + (2-3k)r^2 + ((4a-24)k + 4k^2)r$ $+72k^2 + 32ak - 8$

And now for $i = 0$:

Case	$r \mod 4$	Conditions	$16k^2 Z(0)$
1	2	$a \geq r$	$(2-3k)r^2 + (4ak - 8k^2)r - 4k^3 + 8ak^2 - 8$
2	2	$a \leq r$	$(2-3k)r^2 + (4ak + 8k^2)r - 4k^3 - 8ak^2 - 8$
3	0	$a \geq r-2$	$(2-3k)r^2 + (4ak - 4k^2)r + 8k^2 - 8$
4	0	$a \leq r-2$	$(2-3k)r^2 + (4ak + 12k^2)r - (16a+24)k^2 - 8$

This completes Step Two. At this stage we are ready to attack the proof (proper) of our intended result. We remark that since all the above entries must be zero, we can divide them all by any common factors (such as $4k$), and we still have equality with zero. These reduced versions are what we will often use.

Proposition 6.13. *If a is even, then no ℓ exists which ensures that $Z(r) = Z(0) = 0$.*

Proof. The idea is to show that none of the $Z(r) = 0$ equations α is compatible with any of the $Z(0) = 0$ equations β. If both the α and β equations are satisfied, then we should have

$$Z(r) \pm Z(0) = 0$$

Thus we must compare $\alpha = A, B, C$ with $\beta = 1, 2$ (6 combinations), as well as $\alpha = D, E, F$ with $\beta = 3, 4$ (6 more combinations. In each case, both of the new equations generally involve r, A, a, and so obtaining contradictions can be hard, but the following method is useful in a large number of cases.

We perform the following steps:

(1) Cancel sufficient common factors from all the terms;
(2) Substitute $A = \frac{r^2-4}{k}$;
(3) Solve the $Z(r) + Z(0) = 0$ equation for a (which is not hard as they are all linear in a) and substitute it into the $Z(r) - Z(0) = 0$ equation, taking care to observe that the coefficient of a in $Z(r) + Z(0) = 0$ is non-zero (so there are no "divide by zero" issues);

(4) This gives us a new equation $f_{\alpha\beta}(r) = 0$ to be satisfied, and we then find an argument to prove that the function $f_{\alpha,\beta}$ is positive or negative over the range $2, 4 < r < k$ (choice of 2 or 4 according to which is the minimum value of r).
(5) Hence, we conclude that α, β are not compatible.

We illustrate the procedure once, then just summarise the relevant $f_{\alpha,\beta}$. Take $\alpha = A$ and $\beta = 1$. Cancelling terms, we obtain:

$$\begin{aligned} Z(r) + Z(0) = 0 &= (24 + k + 2)A + ((4a - 12) - 8k)r - 4k^2 + 8(a + 1)k + 16a - 12 \\ Z(r) - Z(0) = 0 &= (r + 2k)A - 6r + 4k + 8a \end{aligned}$$

and now observe that the coefficient of a in the first equation is non-zero, so solving for a and substituting directly into the second, along with $A = \frac{r^2-4}{k}$, we find that we in fact require

$$f_{A,1}(r) = r^4 + 4kr^3 + (4k^2 - 8)r^2 + (8k^2 - 16k)r + 16k^3 - 16k^2 + 16 = 0$$

Since $k \geq 5$, the coefficients are all positive, whence $f_{A,1}(r) > 0$ on $0 < r < k$, giving the contradiction we require.

In a similar vein, we now summarise the other data in the following table.

α, β	$f_{\alpha,\beta}(r)$
$A, 1$	$r^4 + 4kr^3 + (4k^2 - 8)r^2 + (8k^2 - 16k)r + 16k^3 - 16k^2 + 16$
$B, 1$	$r^4 - (5k^2 - 2k + 8)r^2 + 4k^4 + 16k^2 - 8k + 16$
$C, 1$	$r^4 - (3k^2 - 2k + 8)r^2 + 16k^2r - 4k^4 + 32k^3 + 16k^2 - 8k + 16$
$A, 2$	$r^4 - (5k^2 + 2k + 8)r^2 + 4k^4 + 16k^2 + 8k + 16$
$B, 2$	$r^4 - 4kr^3 + (2k^2 - 8)r^2 + (8k^3 - 8k^2 + 16k)r - 8k^4 + 16k^3 - 16k^2 + 16$
$C, 2$	$r^4 - 4kr^3 + (4k^2 - 8)r^2 + (8k^2 + 16k)r - 16k^3 - 16k^2 + 16$
$D, 3$	$r^4 - 8r^2 + 8k^2r - 16k^2 + 16$
$E, 3$	$r^4 - 4kr^3 + (k^2 + 2k - 8)r^2 - (4k^3 - 8k^2 - 16k)r + 8k^3 - 16k^2 - 8k + 16$
$F, 3$	$r^4 + (2k^2 - 8)r^2 + 24k^2r - 16k^2 + 16$
$D, 4$	$r^4 - 4kr^3 - (k^2 + 2k + 8)r^2 + (4k^3 + 8k^2 + 16k)r - 8k^3 + 48k^2 + 8k + 16$
$E, 4$	$r^4 - 8kr^3 + (16k^2 - 8)r^2 + (8k^2 + 32k)r - 32k^3 - 16k^2 + 16$
$F, 4$	$r^4 - 4kr^3 + (k^2 - 2k - 8)r^2 - (4k^3 - 24k^2 - 16k)r - 72k^3 + 48k^2 + 8k + 16$

We now attack these case by case.

A1: Already done.

B1,A2: In both situations, $f_{\alpha,\beta} = r^4 - Nr^2 + M$. The turning points of this quartic occur when $r = 0$ or $r^2 = \frac{N}{2}$, so provided that $\frac{N}{2} \geq k^2$, we know that $f_{\alpha,\beta}$ is decreasing on $0 < r < k$. As this happens to be true, and

$$f_{\alpha,\beta}(r) = \begin{cases} 8k^3 + 5k^2 + 6k + 9 & \alpha = B, \beta = 1 \\ 4k^3 + 13k^2 + 18k + 9 & \alpha = A, \beta = 2 \end{cases}$$

we see that $f_{\alpha,\beta}(r) > 0$ on $0 < r < k$, which is our contradiction.

C1: The function is not obviously useful, but we know $1 + \frac{r}{2} - \frac{a}{2} - \frac{A}{2} \geq 0$ (by case C) and $a \geq r$ (by case 1), whence we are forced to conclude that $A = 0$. However, then $r = 2$ and $a = \frac{k+3}{2}$ by direct computation, and these do not satisfy the condition from case C. Contradiction.

B2: We aim to show that $f(r) := f_{B,2}(r) < 0$ on $0 < r < k$ and for $k \geq 7$. Indeed, compute the derivatives:

$$\begin{aligned} \frac{df}{dr}(r) &= 4r^3 - 12kr^2 + (4k^2 - 16)r + (8k^3 - 8k^2 + 16k) \\ \frac{d^2 f}{dr^2}(r) &= 12r^2 - 24kr + (4k^2 - 16) \\ \frac{d^3 f}{dr^3}(r) &= 24r - 24k \end{aligned}$$

As we can see, $\frac{d^3 f}{dr^3}(r) < 0$, whence $\frac{d^2 f}{dr^2}$ is decreasing. Observing that $\frac{d^2 f}{dr^2}(0) = 4k^2 - 16 > 0$ while $\frac{d^2 f}{dr^2}(k) = -8k^2 - 16 < 0$, we know there is precisely one zero in the range $0 < r < k$. Hence, $\frac{df}{dr}$

has one turning point, and it is a maximum by the negativity of $\frac{d^3 f}{dr^3}$. Checking at both extremes of the range again finds that $\frac{df}{dr}(r) > 0$, and so f is increasing. However,

$$f(k) = -k^4 + 8k^3 - 8k^2 + 16$$

which is negative for $k \geq 7$, and so $f_{B,2}(r) = f(r) < 0$ on the range prescribed. If $k = 5$, then observe that $r = 2$, and direct computation finds $f_{B,2}(2) < 0$.

C2: We play around with the $Z(r) - Z(0) = 0$ equation, which gives

$$8a - (A-6)(2k-r) = 0$$

and rearranged this tells us that

$$2k = r + \tfrac{8a}{A-6}$$

Ponder this a moment. Since $\frac{r}{2}$ is odd, we know that $\frac{r^2}{4} = \frac{A}{4}k + 1 \equiv 1 \mod 4$, and so $A \equiv 0 \mod 16$. If $A \geq 16$, then it must follow that $2k \leq r + \frac{4}{5}r < 2k$, which is nonsense. If $A = 0$, then we find instead $2k = r - \frac{4}{3}a < 2k$, also a contradiction.

D3: Consider $f_{D,3}(r) = (r^4 - 8r^2) + (8k^2 r - 16k^2 + 16)$. The two bracketed expressions are both positive once $r \geq 4$, but since $\frac{r}{2}$ is even (from cases D and 3), it follows $r \equiv 0 \mod 4$, and so $r = 4$ is the smallest value for r allowed. Hence we have our contradiction.

E3: From condition 3 we know that $B = \frac{r}{2} - \frac{a}{2} - \frac{A}{2} \leq 1 - \frac{A}{2} < 0$ unless $A = 0$. This contradicts condition E. However, if $A = 0$, then $r = 2$, $a = \frac{k+3}{2}$, and we violate the condition that $r \equiv 0 \mod 4$.

F3: All the coefficients in $f_{F,3}$ are obviously positive for $k \geq 5$, with the exception of the constant term, but $24k^2 r - 16k^2 > 0$, and so we are done.

D4: Write

$$f_{D,4}(r) = \underbrace{(r^4 - 4kr^3 - k^2r^2 + 4k^3 r - 8k^3)}_{g(r)} + (8k^2 r - 2kr^2) + (48k^2 - 8r^2) + 16kr + 8k + 16$$

and observe that except possibly $g(r)$, all the terms are positive. We aim to show that on the range $4 < r < k$ we have $g(r) > 0$. Indeed, consider its derivatives:

$$\frac{dg}{dr}(r) = 4r^3 - 12kr^2 - 2k^2 r + 4k^3$$
$$\frac{d^2 g}{dr^2}(r) = 12r^2 - 24kr - 2k^2$$

Now, the second derivative is clearly negative on $0 < r < k$, and thus on our range of interest $\frac{dg}{dr}$ is decreasing. Observing that $\frac{dg}{dr}(0) = 4k^3 > 0$ and $\frac{dg}{dr}(k) = -6k^3 < 0$ we know there is precisely one zero to $\frac{dg}{dr}$ on $0 < r < k$. That is, g has precisely one turning point, and since we know that the second derivative is negative we know that this turning point is a local maximum. We compute:

$$g(4) = 8k^3 - 16k^2 - 256k + 256 \qquad g(k-2) = 4k^3 - 28k^2 + 16$$

When $k \geq 7$, these are both positive, so the function is positive over the range $4 < r < k-2$. Notice that the requirements that $r \equiv 0 \mod 4$ and $r^2 = 4 \mod k$ both imply that we need not consider $r = 2, k-1$, and so this suffices for our contradiction. If $k = 5$, we do the usual calculation with $r = 2$, $A = 0$, $a = 4$, and cannot be in this case since condition 4 is violated.

E4: Rearrange the $Z(r) - Z(0) = 0$ equation to obtain

$$4k = r - \tfrac{4}{A-2}(r - 2a)$$

and at this point we know (from condition 4) that $a \leq r - 2$, whence $4k \leq r + \frac{4k}{A-2} < 3k$ if $A \neq 0$, since $A \equiv 0 \mod 4$. If $A = 0$, then $a = \frac{k+3}{2} > r - 2 = 0$, a contradiction.

F4: We rewrite $f_{F,4}$ as:

$$\begin{aligned} f_{F,4}(r) = (r^4 + k^2r^2 - 2k^3 r) - 4kr^3 - (2k+8)r^2 \\ - (2k^3 - 24k^2 - 16k)r - (72k^3 - 48k^2 - 8k - 16) \end{aligned}$$

Once $k \geq 13$, the terms after the first three are all negative. We claim that the first bracketed expression is also negative. Indeed, $r^4 + k^2r^2 < k^3 r + k^3 r = 2k^3 r$, so all of $f_{F,4}(r) < 0$ on the range required. For $k < 13$, we use the usual contradiction since k is prime power.

With all possibilities checked, we are finished the proof. $\square$

6.6. **The case a odd.** We now repeat for a an odd integer. This is extremely similar to the previous situation, so we omit proofs which are virtually identical. We mimic the previous initial propositions, and again proof is just straightforward verification.

Proposition 6.14 ($r \equiv 2 \mod 4$)**.** *If $r \equiv 2 \mod 4$ and a is odd, set $B := 1 + \frac{r}{2} - \frac{a-k}{2} - \frac{A}{2} \in [0, k)$ and $J := -\frac{r}{2} - 4 < 0$, giving*

$$\phi(2s\ell - r\ell) = \begin{cases} -K^1_{B,-J-2B} & J + 2B < k \\ -K^1_{B+2,-J-2B+2k-4} & J + 2B \geq k \end{cases}$$

and also

$$\phi(2s\ell) = \begin{cases} K^1_{\frac{a-r+k}{2},\frac{r}{2}-a-k} & \text{if } r > 2a \\ K^1_{\frac{a-r+k}{2}+2,\frac{r}{2}-a+k-4} & \text{if } r \leq 2a \end{cases}$$

Proposition 6.15 ($r \equiv 0 \mod 4$)**.** *If, on the other hand, $r \equiv 0 \mod 4$, set $B := \frac{r}{2} - \frac{a-k}{2} - \frac{A}{2} \in [0, k)$ and $J := k - \frac{r}{2} - 4 > 0$, giving*

$$\phi(2s\ell - r\ell) = \begin{cases} -K^1_{B,-J-2B} & J + 2B < k \\ -K^1_{B+2,-J-2B+2k-4} & J + 2B \geq k \end{cases}$$

and the beautifully simple

$$\phi(2s\ell) = K^1_{\frac{a-r+k}{2}+1,\frac{r}{2}-a-2}$$

With these computed, we then establish the tables exactly as before. First, $i = 0$:

Case	$r \mod 4$	Conditions	$16k^2 Z(0)$
1	2	$r > 2a$	$(2-3k)r^2 + (4akr - 4k^2)r + 8ak^2 + 4k^3 - 8$
2	2	$r \leq 2a$	$(2-3k)r^2 + (4akr + 4k^2)r - 8ak^2 + 4k^3 - 8$
3	0	$-$	$(2-3k)r^2 + 4akr + 8k^2 - 8$

And now for $i = r$:

Case	$r \mod 4$	Conditions	$16k^2 Z(r)$
A	2	$J + 2B < k$	$(4kr - 8k^2)A + (2-3k)r^2 + ((4a-24)k + 4k^2)r + 4k^3 + (-8a+16)k^2 + 32ak - 8$
B	2	$J + 2B \geq k$	$(4kr - 24k^2)A + (2-3k)r^2 + ((4a-24)k + 12k^2)r + 36k^3 + (-24a-80)k^2 + 32ak - 8$
C	0	$J + 2B < k$	$(4kr - 16k^2)A + (2-3k)r^2 + ((4a-24)k + 8k^2)r + 16k^3 + (-16a-24)k^2 + 32ak - 8$
D	0	$J + 2B \geq k$	$4Akr + (2-3k)r^2 + (4a-24)kr + 40k^2 + 32ak - 8$

And thus we can mimic the previous proof to obtain the following proposition.

Proposition 6.16. *If a is odd, then no ℓ exists which ensures that $Z(r) = Z(0) = 0$.*

Proof. Exactly as before, we have another table (though it is much smaller this time):

α, β	$f_{\alpha,\beta}(r)$
$A, 1$	$r^4 - (5k^2 - 2k + 8)r^2 + 4k^4 + 16k^2 - 8k + 16$
$A, 2$	$r^4 - 4kr^3 + (4k^2 - 8)r^2 + (8k^2 + 16k)r - 16k^3 - 16k^2 + 16$
$B, 1$	$r^4 - 4kr^3 - (14k^2 - 4k + 8)r^2 + (8k^3 - 24k^2 + 16k)r + 24k^4 - 48k^3 + 48k^2 - 16k + 16$
$B, 2$	$r^4 - 8kr^3 + (11k^2 + 2k - 8)r^2 + (8k^3 - 16k^2 + 32k)r - 12k^4 + 32k^3 - 48k^2 - 8k + 16$
$C, 3$	$r^4 - 4kr^3 - (k^2 - 2k + 8)r^2 + (4k^3 - 8k^2 + 16k)r + 8k^3 - 16k^2 - 8k + 16$
$D, 3$	$r^4 - 8r^2 + 8k^2 r - 16k^2 + 16$

And the case-by-case analysis goes as follows.

- **A1:** We observe that $f_{A,1}$ has the same structure as cases A2 and B1 from the previous section, and that $f_{A,1} = 8k^3 + 5k^2 + 6k + 9 > 0$, whence we are done.

A2: We observe the $Z(r) - Z(0) = 0$ equation, which gives us
$$k = \frac{4a - 2r}{A - 2} + \frac{r}{2}$$
and since $A \equiv 0 \mod 16$ (see case C2 in the previous section), barring $A = 0$, we discover
$$k < \tfrac{1}{7}(2a - r) + \tfrac{r}{2} < \tfrac{5}{7}k$$
which is a contradiction. If $A = 0$, we see $k = r - 2a + \frac{r}{2} < k$ (since $r \leq 2a$ by condition 2), also a contradiction.

B1: We write the function as
$$f_{B,1}(r) = r^4 + (4k^4 - 4kr^3) + (14k^4 - (14k^2 - 4k + 8)r^2) \\ + (8k^3 - 24k^2 + 16k)r + (6k^4 - 48k^3 + 48k^2 - 16k + 16)$$
and observe that each bracketed term is positive for $k \geq 7$. For $k = 5$, we use $r = 2$, $A = 0$, $a = 4$, and this is forbidden since a should be odd.

B2: From condition B, we see $J + 2B \geq k$, so $\frac{r}{2} - a - A - 2 \geq 0$. However, from condition 2, we know that $r \leq 2a$, so we have a contradiction.

C3: We write
$$f_{C,3} = g(r) + (2k - 8)r^3 + (8k^3 - 8k^2 r + 16kr) + (8k^3 - 16k^2 - 8k + 16)$$
where $g(r)$ is the same function as in case D4 above. We know that all terms are positive for $k \geq 7$, and if $k = 5$ we obtain the usual contradiction (namely, we are not in this case).

D3: We write
$$f_{D,3}(r) = (r^4 - 8r^2) + (8k^2 r - 16k^2) + 16$$
and as soon $r \geq 4$ we see that both bracketed terms are positive. Since $r \equiv 0 \mod 4$, we are done.

All cases are done, and so is the proof. ☐

6.7. **The proof.** We can finally complete the slice case.

Theorem 6.17. *If $P(k, -k, 2m)$ has unknotting number one, with $k > 0$ odd and $m > 0$ even, then $k = 3$ and $m = 1$, up to reflection.*

Proof. For any m, we know $k = 1$ yields the unknot. Otherwise, we know by Lemma 5.3 that $m = 1$. Moreover, we now know from the previous two subsections that if $k \geq 5$ then $P(k, -k, 2)$ cannot have unknotting number one. $P(3, -3, 2)$ does indeed have unknotting number one, and the theorem is proved. ☐

6.8. **Examples of Symmetry or Asymmetry.** We decompose the above in the case that k is prime power. Recall from Proposition 6.9, there is an essentially unique ℓ. Then a is even or odd according to the congruence of k modulo 4, but in either situation we are in case A1. We get:
$$\phi(2\ell) = K^3_{k-4} \qquad\qquad \phi((2s-2)\ell) = \begin{cases} -K^3_k & k = 5 \\ K^1_{\frac{1}{4}(k-5), -\frac{1}{2}(k+5)} & k > 5 \text{ and } k \equiv 1 \mod 4 \\ -K^1_{\frac{1}{4}(k+5), -\frac{1}{2}(k-5)} & k \equiv 3 \mod 4 \end{cases}$$
We then find (surprisingly independently of the conditions on k modulo 4):
$$d(\Sigma, \phi(2\ell)) = -\tfrac{1}{k^2}(-2k^2 + 8) \qquad\qquad d(\Sigma, \phi((2s-2)\ell)) = -\tfrac{1}{2k^2}(-k^2 + 25)$$
Grinding all this into the Ozsváth-Szabó unknotting theorem, we should find $Z(2) = 0$, but in fact:
$$Z(2) = \tfrac{1}{2k^2}(3k^2 + 9) + \tfrac{1}{2k^2}(k^2 - 9) = 2$$
which is blatantly untrue.

We can see this even more concretely in a particular example, namely $k = 5$. We can compute that the correction terms for the lens space are
$$d(L, i) = (0, -\tfrac{2}{25}, -\tfrac{8}{25}, -\tfrac{18}{25}, -\tfrac{32}{25}, -2, -\tfrac{72}{25}, -\tfrac{48}{25}, -\tfrac{28}{25}, -\tfrac{12}{25}, 0, \tfrac{8}{25}, \tfrac{12}{25}, \dots)$$
Here, we have only presented the first half, since $d(\cdot, i) = d(\cdot, -i)$. Then for the double cover, we have, using our isomorphism ϕ,
$$d(\Sigma, i') = (0, \tfrac{22}{25}, -\tfrac{12}{25}, -\tfrac{2}{25}, \tfrac{2}{25}, 0, \tfrac{42}{25}, \tfrac{28}{25}, \tfrac{8}{25}, -\tfrac{18}{25}, 0, \tfrac{12}{25}, \tfrac{18}{25}, \dots)$$

Now, solving the congruence from our symmetry theorem tells us $\ell = \pm 3$, so take $\ell = 22$ (note that indeed $r = 2$, $A = 0$, and $a = 4$).

$$d(\Sigma, 22i') = (0, -\tfrac{2}{25}, \tfrac{42}{25}, -\tfrac{18}{25}, \tfrac{18}{25}, 0, \tfrac{28}{25}, \tfrac{2}{25}, \tfrac{22}{25}, -\tfrac{12}{25}, 0, \tfrac{8}{25}, \tfrac{12}{25}, \dots)$$

Now we compute the corresponding sides of the Ozsváth-Szabó unknotting theorem, and multiply them by 25:

i	$\Sigma(k,-k,2)$	$-L(k^2,2)$
0	-12	-12
1	-10	-10
2	42	-8
3	-6	-6
4	-4	-4
5	-2	-2
6	0	0

We can see here that the two sides are congruent modulo 25, but not equal, so the knot, which is $12n_{721}$, cannot have unknotting number one. We can also explicitly see the failure of $Z(2) = 0$, and the correct value is indeed $Z(2) = 2$.

We conclude our exploration of the slice case with a few remarks on the symmetry in the pretzel $P(3,-3,2)$, since it actually has unknotting number one. First, we establish a group structure, which, as presented in this paper, is exactly

$$\phi : \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} \longmapsto \begin{pmatrix} K_1^2 & K_{1,-1}^1 & K_{3,-1}^1 \\ K_3^3 & K_1^3 & K_{-1}^3 \\ K_{-3}^3 & K_{2,-3}^1 & K_{2,-5}^1 \end{pmatrix}$$

Now, it is trivial to compute $s = 2$, whence the symmetry theorem is really only interesting for $i = 0, 1$. Using our congruence, we know that $\ell = \pm 2$, and then we then find that the proper isomorphism to choose is

$$\widetilde{\phi} : \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} \longmapsto \begin{pmatrix} K_1^2 & K_{3,-1}^1 & K_1^3 \\ K_{-3}^3 & K_{2,-5}^1 & K_{1,-1}^1 \\ K_3^3 & K_{-1}^3 & K_{2,-3}^1 \end{pmatrix}$$

Hence, it is straightforward to check that Ozsváth and Szabó's unknotting theorem is satisfied.

References

[1] Steven A. Bleiler. A note on unknotting number. *Math. Proc. Cambridge Philos. Soc.*, 96(3):469–471, 1984.

[2] Abhijit Champanerkar and Ilya Kofman. Twisting quasi-alternating links. *Proceedings of the American Mathematical Society*, 137(7):2451 – 2458, 2009.

[3] S. K. Donaldson. The orientation of Yang-Mills moduli spaces and 4-manifold topology. *J. Differential Geom.*, 26(3):397–428, 1987.

[4] C. McA. Gordon and R. A. Litherland. On the signature of a link. *Invent. Math.*, 47(1):53–69, 1978.

[5] C. McA. Gordon and John Luecke. Knots with unknotting number 1 and essential Conway spheres. *Algebr. Geom. Topol.*, 6:2051–2116 (electronic), 2006.

[6] Joshua E. Greene. On closed 3-braids with unknotting number one. *arXiv:0902.1573v1 [math.GT]*, 2009.

[7] Joshua E. Greene and Stanislav Jabuka. The slice-ribbon conjecture for 3-stranded pretzel knots. *arXiv:0706.3398v2 [math.GT]*, 2007.

[8] Taizo Kanenobu and Hitoshi Murakami. Two-bridge knots with unknotting number one. *Proc. Amer. Math. Soc.*, 98(3):499–502, 1986.

[9] Tsuyoshi Kobayashi. Minimal genus Seifert surfaces for unknotting number 1 knots. *Kobe J. Math.*, 6(1):53–62, 1989.

[10] P. B. Kronheimer and T. S. Mrowka. Gauge theory for embedded surfaces. *Topology*, 32:773,826, 1993.

[11] Ana Lecuona. Private communication concerning upcoming paper joint with Stanislav Jabuka. 2011.

[12] W. B. Raymond Lickorish. *An Introduction to Knot Theory.* Number 175 in Graduate Texts in Mathematics. Springer-Verlag, 1991.

[13] José M. Montesinos. Surgery on links and double branched covers of S^3. In *Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox)*, pages 227–259. Ann. of Math. Studies, No. 84. Princeton Univ. Press, Princeton, N.J., 1975.

[14] Louise Moser. Elementary surgery along a torus knot. *Pacific Journal of Mathematics*, 38(3):737 – 745, 1971.

[15] Kunio Murasugi. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, 117:387–422, 1965.

[16] Yasutaka Nakanishi. A note on unknotting number. *Math. Sem. Notes Kobe Univ.*, 9:99 – 108, 1981.

[17] Yasutaka Nakanishi. Unknotting numbers and knot diagrams with the minimum crossings. *Math. Sem. Notes Kobe Univ.*, 11(2):257–258, 1983.

[18] Walter D. Neumann and Frank Raymond. Seifert manifolds, plumbing, μ-invariant and orientation reversing maps. In *Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977)*, volume 664 of *Lecture Notes in Math.*, pages 163–196. Springer, Berlin, 1978.

[19] Peter Ozsváth and Zoltán Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Advances in Mathematics*, 173:179 – 261, 2003.

[20] Peter Ozsváth and Zoltán Szabó. On the Floer homology of plumbed three-manifolds. *Geom. Topol.*, 7:185–224 (electronic), 2003.

[21] Peter Ozsváth and Zoltán Szabó. Knots with unknotting number one and Heegaard Floer homology. *Topology*, 44(4):705–745, 2005.

[22] Peter Ozsváth and Zoltán Szabó. On knot Floer homology and lens space surgeries. *Topology*, 44(6):1281 – 1300, 2005.

[23] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

[24] Martin Scharlemann and Abigail Thompson. Link genus and the Conway moves. *Comment. Math. Helv.*, 64(4):527–535, 1989.

[25] Martin G. Scharlemann. Unknotting number one knots are prime. *Inventiones Mathematicae*, 82:37 – 55, 1985.

[26] A. Stoimenow. Some examples related to 4-genera, unknotting numbers and knot polynomials. *J. London Math. Soc. (2)*, 63(2):487–500, 2001.

[27] Morwen B. Thistlethwaite. On the algebraic part of an alternating link. *Pacific J. Math.*, 151(2):317–333, 1991.

[28] Ichiro Torisu. A note on Montesinos links with unlinking number one (conjectures and partial solutions). *Kobe J. Math.*, 13(2):167–175, 1996.

[29] Xingru Zhang. Unknotting number one knots are prime: A new proof. *Proceedings of the American Mathematical Society*, 113(2):611 – 612, 1991.